A TWO WEIGHT THEOREM FOR α-FRACTIONAL SINGULAR INTEGRALS WITH AN ENERGY SIDE CONDITION, QUASICUBE TESTING AND COMMON POINT MASSES

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Abstract. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$ (possibly having common point masses), and let T^α be a standard α-fractional Calderón-Zygmund operator on $\mathbb{R}^n$ with $0 \leq \alpha < n$. Suppose that $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map, and refer to the images ΩQ of cubes Q as *quasicubes*. Furthermore, assume as side conditions the $\mathcal{A}_2^\alpha$ conditions, punctured A_2^α conditions, and certain α-*energy conditions* taken over quasicubes. Then we show that T^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if the quasicube testing conditions hold for T^α and its dual, and if the quasiweak boundedness property holds for T^α.

Conversely, if T^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$, then the quasitesting conditions hold, and the quasiweak boundedness condition holds. If the vector of α-fractional Riesz transforms $\mathbf{R}_\sigma^\alpha$ (or more generally a strongly elliptic vector of transforms) is bounded from $L^2(\sigma)$ to $L^2(\omega)$, then both the $\mathcal{A}_2^\alpha$ conditions and the punctured A_2^α conditions hold. We do not know if our quasienergy conditions are necessary when $n \geq 2$, except for certain situations in which one of the measures is one-dimensional, or both measures are sufficiently dispersed.

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Date: April 16, 2016.
Research supported in part by NSERC.
C.-Y. Shen supported in part by the NSC, through grant NSC102-2115-M-008-015-MY2.
I. Uriarte-Tuero has been partially supported by grants DMS-1056965 (US NSF), MTM2010-16232, MTM2015-65792-P (MINECO, Spain), and a Sloan Foundation Fellowship.

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1. Introduction

In this paper we prove a two weight inequality for standard α-fractional Calderón-Zygmund operators T^{α} in Euclidean space $\mathbb{R}^n$, where we assume n-dimensional $\mathcal{A}_2^{\alpha}$ conditions (with holes), punctured $A_2^{\alpha,\mathrm{punct}}$ conditions, and certain *α-energy conditions* as side conditions on the weights (in higher dimensions the Poisson kernels used in these two conditions differ). We state and prove our theorem in the more general setting of *quasicubes* as in [SaShUr5], but here we now permit the weights, or measures, to have common point masses, something not permitted in [SaShUr5]. As a consequence, we use $\mathcal{A}_2^{\alpha}$ conditions with holes as in the one-dimensional setting of Hytönen [Hyt2], together with punctured $A_2^{\alpha,\mathrm{punct}}$ conditions, as the usual A_2^{α} '*without punctures*' fails whenever the measures have a common point mass. The extension to permitting common point masses uses the two weight Poisson inequality in [Saw] to derive functional energy, together with a delicate adaptation of arguments in [SaShUr5]. The key point here is the use of the (typically necessary) 'punctured' Muckenhoupt $A_2^{\alpha,\mathrm{punct}}$ conditions below. They turn out to be crucial in estimating the Poisson testing conditions later in the paper. We remark that Hytönen's bilinear dyadic Poisson operator and shifted dyadic grids ([Hyt2]) in dimension $n=1$ can be extended to derive functional energy in higher dimensions, but at a significant cost of increased complexity. See the earlier versions of this paper on the *arXiv* for this approach[1], and also [LaWi] where Lacey and Wick use this approach. Finally, we point out that our use of punctured Muckenhoupt conditions provides a simpler alternative to Hytönen's method of extending to common point masses the NTV conjecture for the Hilbert transform [Hyt2].

On the other hand, the extension to quasicubes in the setting of common point masses turns out to be, after checking all the details, mostly a cosmetic modification of the proof in [SaShUr5], except for the derivation of the n-dimensional $\mathcal{A}_2^{\alpha}$ conditions with holes, which requires extensive modification of earlier arguments.

We begin by recalling the notion of quasicube used in [SaShUr5] - a special case of the classical notion used in quasiconformal theory.

Definition 1. *We say that a homeomorphism $\Omega:\mathbb{R}^n\to\mathbb{R}^n$ is a globally biLipschitz map if*

$$\left\|\Omega\right\|_{Lip}\equiv\sup_{x,y\in\mathbb{R}^n}\frac{\left\|\Omega\left(x\right)-\Omega\left(y\right)\right\|}{\left\|x-y\right\|}<\infty, \tag{1.1}$$

and $\left\|\Omega^{-1}\right\|_{Lip}<\infty$.

Note that a globally biLipschitz map Ω is differentiable almost everywhere, and that there are constants $c,C>0$ such that

$$c\leq J_{\Omega}\left(x\right)\equiv\left|\det D\Omega\left(x\right)\right|\leq C,\qquad x\in\mathbb{R}^n.$$

Example 1. *Quasicubes can be wildly shaped, as illustrated by the standard example of a logarithmic spiral in the plane $f_{\varepsilon}\left(z\right)=z\left|z\right|^{2\varepsilon i}=ze^{i\varepsilon\ln\left(z\overline{z}\right)}$. Indeed, $f_{\varepsilon}:\mathbb{C}\to\mathbb{C}$ is a globally biLipschitz map with Lipschitz constant $1+C\varepsilon$ since $f_{\varepsilon}^{-1}\left(w\right)=w\left|w\right|^{-2\varepsilon i}$ and*

$$\nabla f_{\varepsilon}=\left(\frac{\partial f_{\varepsilon}}{\partial z},\frac{\partial f_{\varepsilon}}{\partial\overline{z}}\right)=\left(\left|z\right|^{2\varepsilon i}+i\varepsilon\left|z\right|^{2\varepsilon i},i\varepsilon\frac{z}{\overline{z}}\left|z\right|^{2\varepsilon i}\right).$$

[1]Additional small arguments are needed to complete the shifted dyadic proof given there, but we omit them in favour of the simpler approach here resting on punctured Muckenhoupt conditions instead of holes. The authors can be contacted regarding completion of the shifted dyadic proof.

On the other hand, f_ε behaves wildly at the origin since the image of the closed unit interval on the real line under f_ε is an infinite logarithmic spiral.

Notation 1. *We define $\mathcal{P}^n$ to be the collection of half open, half closed cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. A half open, half closed cube Q in $\mathbb{R}^n$ has the form $Q = Q(c,\ell) \equiv \prod_{k=1}^{n} \left[c_k - \frac{\ell}{2}, c_k + \frac{\ell}{2}\right)$ for some $\ell > 0$ and $c = (c_1, ..., c_n) \in \mathbb{R}^n$. The cube $Q(c,\ell)$ is described as having center c and sidelength ℓ.*

We typically use $\mathcal{D}$ to denote a dyadic grid of cubes from $\mathcal{P}^n$. We repeat the natural *quasi* definitions from [SaShUr5].

Definition 2. *Suppose that $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map.*

(1) *If E is a measurable subset of $\mathbb{R}^n$, we define $\Omega E \equiv \{\Omega(x) : x \in E\}$ to be the image of E under the homeomorphism Ω.*
   (a) *In the special case that $E = Q$ is a cube in $\mathbb{R}^n$, we will refer to ΩQ as a quasicube (or Ω-quasicube if Ω is not clear from the context).*
   (b) *We define the center $c_{\Omega Q} = c(\Omega Q)$ of the quasicube ΩQ to be the point Ωc_Q where $c_Q = c(Q)$ is the center of Q.*
   (c) *We define the side length $\ell(\Omega Q)$ of the quasicube ΩQ to be the sidelength $\ell(Q)$ of the cube Q.*
   (d) *For $r > 0$ we define the 'dilation' $r\Omega Q$ of a quasicube ΩQ to be ΩrQ where rQ is the usual 'dilation' of a cube in $\mathbb{R}^n$ that is concentric with Q and having side length $r\ell(Q)$.*

(2) *If $\mathcal{K}$ is a collection of cubes in $\mathbb{R}^n$, we define $\Omega\mathcal{K} \equiv \{\Omega Q : Q \in \mathcal{K}\}$ to be the collection of quasicubes ΩQ as Q ranges over $\mathcal{K}$.*

(3) *If $\mathcal{F}$ is a grid of cubes in $\mathbb{R}^n$, we define the inherited quasigrid structure on $\Omega\mathcal{F}$ by declaring that ΩQ is a child of $\Omega Q'$ in $\Omega\mathcal{F}$ if Q is a child of Q' in the grid $\mathcal{F}$. We denote by $\mathfrak{C}(Q)$ the collection of children of Q.*

Note that if ΩQ is a quasicube, then $|\Omega Q|^{\frac{1}{n}} \approx |Q|^{\frac{1}{n}} = \ell(Q) = \ell(\Omega Q)$ shows that the measure of ΩQ is approximately its sidelength to the power n, more precisely there are positive constants c, C such that $c|J|^{\frac{1}{n}} \leq \ell(J) \leq C|J|^{\frac{1}{n}}$ for any quasicube $J = \Omega Q$. We will generally use the expression $|J|^{\frac{1}{n}}$ in the various estimates arising in the proofs below, but will often use $\ell(J)$ when defining collections of quasicubes. Moreover, there are constants R_{big} and R_{small} such that we have the comparability containments

$$Q + \Omega x_Q \subset R_{big}\Omega Q \text{ and } R_{small}\Omega Q \subset Q + \Omega x_Q \ .$$

Given a fixed globally biLipschitz map Ω on $\mathbb{R}^n$, we will define below the n-dimensional $\mathcal{A}_2^\alpha$ conditions (with holes), punctured Muckenhoupt conditions $A_2^{\alpha,\mathrm{punct}}$, testing conditions, and energy conditions using Ω-quasicubes in place of cubes, and we will refer to these new conditions as quasi$\mathcal{A}_2^\alpha$, quasitesting and quasienergy conditions. We will then prove a $T1$ theorem with quasitesting and with quasi$\mathcal{A}_2^\alpha$ and quasienergy side conditions on the weights. We now describe a particular case informally, and later explain the full theorem in detail.

We show that for positive locally finite Borel measures σ and ω, possibly having common point masses, and *assuming* the quasienergy conditions in the Theorem below, a strongly elliptic collection of standard α-fractional Calderón-Zygmund operators $\mathbf{T}^\alpha$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$,

$$\left\|\mathbf{T}^\alpha(f\sigma)\right\|_{L^2(\omega)} \leq \mathfrak{N}_{\mathbf{T}^\alpha}\left\|f\right\|_{L^2(\sigma)}, \tag{1.2}$$

(with $0 \leq \alpha < n$) *if and only if* the $\mathcal{A}_2^\alpha$ condition and its dual hold (we assume a mild additional condition on the quasicubes for this), the punctured Muckenhoupt condition $A_2^{\alpha,\mathrm{punct}}$ and its dual hold, the quasicube testing condition for $\mathbf{T}^\alpha$ and its dual hold, and the quasiweak boundedness property holds.

Since the $\mathcal{A}_2^\alpha$ and punctured Muckenhoupt conditions typically hold, this identifies the culprit in higher dimensions as the pair of quasienergy conditions. We point out that these quasienergy conditions are implied by higher dimensional analogues of essentially all the other side conditions used previously in two weight theory, in particular doubling conditions and the Energy Hypothesis (1.16) in [LaSaUr2], as well as the uniformly full dimension hypothesis in [LaWi] (uniformly full dimension permits a reversal of energy, something not assumed in this paper, and reversal of energy implies our energy conditions).

It turns out that in higher dimensions, there are two natural 'Poisson integrals' P^α and $\mathcal{P}^\alpha$ that arise, the usual Poisson integral P^α that emerges in connection with energy considerations, and a different Poisson

integral $\mathcal{P}^\alpha$ that emerges in connection with size considerations - in dimension $n = 1$ these two Poisson integrals coincide. The standard Poisson integral P^α appears in the energy conditions, and the reproducing Poisson integral $\mathcal{P}^\alpha$ appears in the $\mathcal{A}_2^\alpha$ condition. These two kernels coincide in dimension $n = 1$ for the case $\alpha = 0$ corresponding to the Hilbert transform.

2. Statements of results

Now we turn to a precise description of our two weight theorem.

Assumption: We fix once and for all a globally biLipschitz map $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ for use in all of our quasi-notions.

As already mentioned above we will prove a two weight inequality for standard α-fractional Calderón-Zygmund operators T^α in Euclidean space $\mathbb{R}^n$, where we assume the n-dimensional $\mathcal{A}_2^\alpha$ conditions, new punctured A_2^α conditions, and certain *α-quasienergy conditions* as side conditions on the weights. In order to state our theorem precisely, we need to define standard fractional singular integrals, the two different Poisson kernels, and a quasienergy condition sufficient for use in the proof of the two weight theorem. These are introduced in the following subsections.

Remark 1. *It is possible to collect our various Muckenhoupt and quasienergy assumptions on the weight pair (σ, ω) into just* two *compact side conditions of Muckenhoupt and quasienergy type. We prefer however, to keep the individual conditions separate so that the interested reader can track their use through the arguments below.*

2.1. **Standard fractional singular integrals and the norm inequality.** Let $0 \leq \alpha < n$. Consider a kernel function $K^\alpha(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following fractional size and smoothness conditions of order $1 + \delta$ for some $\delta > 0$,

$$\begin{aligned}
|K^\alpha (x,y)| &\leq C_{CZ} |x-y|^{\alpha-n}, \\
|\nabla K^\alpha (x,y)| &\leq C_{CZ} |x-y|^{\alpha-n-1}, \\
|\nabla K^\alpha (x,y) - \nabla K^\alpha (x',y)| &\leq C_{CZ} \left(\frac{|x-x'|}{|x-y|}\right)^\delta |x-y|^{\alpha-n-1}, \qquad \frac{|x-x'|}{|x-y|} \leq \frac{1}{2}, \\
|\nabla K^\alpha (x,y) - \nabla K^\alpha (x,y')| &\leq C_{CZ} \left(\frac{|y-y'|}{|x-y|}\right)^\delta |x-y|^{\alpha-n-1}, \qquad \frac{|y-y'|}{|x-y|} \leq \frac{1}{2}.
\end{aligned} \tag{2.1}$$

We note that a more general definition of kernel has only order of smoothness $\delta > 0$, rather than $1 + \delta$, but the use of the Monotonicity and Energy Lemmas below, which involve first order Taylor approximations to the kernel functions $K^\alpha(\cdot, y)$, requires order of smoothness more than 1.

2.1.1. *Defining the operators and norm inequality.* We now turn to a precise definition of the weighted norm inequality

$$\|T_\sigma^\alpha f\|_{L^2(\omega)} \leq \mathfrak{N}_{T_\sigma^\alpha} \|f\|_{L^2(\sigma)}, \qquad f \in L^2(\sigma). \tag{2.2}$$

For this we introduce a family $\left\{\eta_{\delta,R}^\alpha\right\}_{0<\delta<R<\infty}$ of nonnegative functions on $[0, \infty)$ so that the truncated kernels $K_{\delta,R}^\alpha(x, y) = \eta_{\delta,R}^\alpha(|x - y|) K^\alpha(x, y)$ are bounded with compact support for fixed x or y. Then the truncated operators

$$T_{\sigma,\delta,R}^\alpha f(x) \equiv \int_{\mathbb{R}^n} K_{\delta,R}^\alpha (x,y) f(y) \, d\sigma(y), \qquad x \in \mathbb{R}^n,$$

are pointwise well-defined, and we will refer to the pair $\left(K^\alpha, \left\{\eta_{\delta,R}^\alpha\right\}_{0<\delta<R<\infty}\right)$ as an α-fractional singular integral operator, which we typically denote by T^α, suppressing the dependence on the truncations.

Definition 3. *We say that an α-fractional singular integral operator $T^\alpha = \left(K^\alpha, \left\{\eta_{\delta,R}^\alpha\right\}_{0<\delta<R<\infty}\right)$ satisfies the norm inequality (2.2) provided*

$$\left\|T_{\sigma,\delta,R}^\alpha f\right\|_{L^2(\omega)} \leq \mathfrak{N}_{T_\sigma^\alpha} \|f\|_{L^2(\sigma)}, \qquad f \in L^2(\sigma), 0 < \delta < R < \infty.$$

It turns out that, in the presence of Muckenhoupt conditions, the norm inequality (2.2) is independent of the choice of appropriate truncations used, and we now explain this in some detail. A *smooth truncation* of T^{α} has kernel $\eta_{\delta,R}\left(|x-y|\right)K^{\alpha}\left(x,y\right)$ for a smooth function $\eta_{\delta,R}$ compactly supported in (δ,R), $0<\delta<R<\infty$, and satisfying standard CZ estimates. A typical example of an α-fractional transform is the α-fractional *Riesz* vector of operators

$$\mathbf{R}^{\alpha,n}=\left\{R_{\ell}^{\alpha,n}:1\leq\ell\leq n\right\},\qquad 0\leq\alpha<n.$$

The Riesz transforms $R_{\ell}^{n,\alpha}$ are convolution fractional singular integrals $R_{\ell}^{n,\alpha}f\equiv K_{\ell}^{n,\alpha}\ast f$ with odd kernel defined by

$$K_{\ell}^{\alpha,n}\left(w\right)\equiv\frac{w^{\ell}}{\left|w\right|^{n+1-\alpha}}\equiv\frac{\Omega_{\ell}\left(w\right)}{\left|w\right|^{n-\alpha}},\qquad w=\left(w^{1},...,w^{n}\right).$$

However, in dealing with energy considerations, and in particular in the Monotonicity Lemma below where first order Taylor approximations are made on the truncated kernels, it is necessary to use the *tangent line truncation* of the Riesz transform $R_{\ell}^{\alpha,n}$ whose kernel is defined to be $\Omega_{\ell}\left(w\right)\psi_{\delta,R}^{\alpha}\left(|w|\right)$ where $\psi_{\delta,R}^{\alpha}$ is continuously differentiable on an interval $(0,S)$ with $0<\delta<R<S$, and where $\psi_{\delta,R}^{\alpha}\left(r\right)=r^{\alpha-n}$ if $\delta\leq r\leq R$, and has constant derivative on both $(0,\delta)$ and (R,S) where $\psi_{\delta,R}^{\alpha}\left(S\right)=0$. Here S is uniquely determined by R and α. Finally we set $\psi_{\delta,R}^{\alpha}\left(0\right)=0$ as well, so that the kernel vanishes on the diagonal and common point masses do not 'see' each other. Note also that the tangent line extension of a $C^{1,\delta}$ function on the line is again $C^{1,\delta}$ with no increase in the $C^{1,\delta}$ norm.

It was shown in the one dimensional case with no common point masses in [LaSaShUr3], that boundedness of the Hilbert transform H with one set of appropriate truncations together with the A_{2}^{α} condition without holes, is equivalent to boundedness of H with any other set of appropriate truncations. We need to extend this to $\mathbf{R}^{\alpha,n}$ and to more general operators in higher dimensions, and also to permit common point masses, so that we are free to use the tangent line truncations throughout the proof of our theorem. For this purpose, we note that the difference between the tangent line truncated kernel $\Omega_{\ell}\left(w\right)\psi_{\delta,R}^{\alpha}\left(|w|\right)$ and the corresponding cutoff kernel $\Omega_{\ell}\left(w\right)\mathbf{1}_{[\delta,R]}\left|w\right|^{\alpha-n}$ satisfies (since both kernels vanish at the origin)

$$\begin{aligned}
&\left|\Omega_{\ell}\left(w\right)\psi_{\delta,R}^{\alpha}\left(|w|\right)-\Omega_{\ell}\left(w\right)\mathbf{1}_{[\delta,R]}\left|w\right|^{\alpha-n}\right|\\
\lesssim&\sum_{k=0}^{\infty}2^{-k(n-\alpha)}\left\{\left(2^{-k}\delta\right)^{\alpha-n}\mathbf{1}_{[2^{-k-1}\delta,2^{-k}\delta]}\left(|w|\right)\right\}+\sum_{k=1}^{\infty}2^{-k(n-\alpha)}\left\{\left(2^{k}R\right)^{\alpha-n}\mathbf{1}_{[2^{k-1}R,2^{k}R]}\left(|w|\right)\right\}\\
\equiv&\sum_{k=0}^{\infty}2^{-k(n-\alpha)}K_{2^{-k}\delta}\left(w\right)+\sum_{k=1}^{\infty}2^{-k(n-\alpha)}K_{2^{k}R}\left(w\right),
\end{aligned}$$

where the kernels $K_{\rho}\left(w\right)\equiv\frac{1}{\rho^{n-\alpha}}\mathbf{1}_{[\rho,2\rho]}\left(|w|\right)$ are easily seen to satisfy, uniformly in ρ, the norm inequality (2.8) with constant controlled by the offset A_{2}^{α} condition (2.3) below. The equivalence of the norm inequality for these two families of truncations now follows from the summability of the series $\sum_{k=0}^{\infty}2^{-k(n-\alpha)}$ for $0\leq\alpha<n$. The case of more general families of truncations and operators is similar.

2.2. **Quasicube testing conditions.** The following 'dual' quasicube testing conditions are necessary for the boundedness of T^{α} from $L^{2}\left(\sigma\right)$ to $L^{2}\left(\omega\right)$, where $\Omega\mathcal{P}^{n}$ denotes the collection of all quasicubes in $\mathbb{R}^{n}$ whose preimages under Ω are usual cubes with sides parallel to the coordinate axes:

$$\begin{aligned}
\mathfrak{T}_{T^{\alpha}}^{2}&\equiv\sup_{Q\in\Omega\mathcal{P}^{n}}\frac{1}{\left|Q\right|_{\sigma}}\int_{Q}\left|T^{\alpha}\left(\mathbf{1}_{Q}\sigma\right)\right|^{2}\omega<\infty,\\
\left(\mathfrak{T}_{T^{\alpha}}^{\ast}\right)^{2}&\equiv\sup_{Q\in\Omega\mathcal{P}^{n}}\frac{1}{\left|Q\right|_{\omega}}\int_{Q}\left|\left(T^{\alpha}\right)^{\ast}\left(\mathbf{1}_{Q}\omega\right)\right|^{2}\sigma<\infty,
\end{aligned}$$

and where we interpret the right sides as holding uniformly over all tangent line truncations of T^{α}.

Remark 2. *We alert the reader that the symbols Q,I,J,K will all be used to denote either cubes or quasicubes, and the context will make clear which is the case. Throughout most of the proof of the main theorem only quasicubes are considered.*

2.3. **Quasiweak boundedness property.** The quasiweak boundedness property for T^{α} with constant C is given by

$$\left|\int_{Q} T^{\alpha}\left(1_{Q'}\sigma\right) d\omega\right| \leq \mathcal{WBP}_{T^{\alpha}}\sqrt{\left|Q\right|_{\omega}\left|Q'\right|_{\sigma}},$$
$$\text{for all quasicubes } Q, Q' \text{ with } \frac{1}{C} \leq \frac{\left|Q\right|^{\frac{1}{n}}}{\left|Q'\right|^{\frac{1}{n}}} \leq C,$$
$$\text{and either } Q \subset 3Q' \setminus Q' \text{ or } Q' \subset 3Q \setminus Q,$$

and where we interpret the left side above as holding uniformly over all tangent line trucations of T^{α}. Note that the quasiweak boundedness property is implied by either the *tripled* quasicube testing condition,

$$\left\|\mathbf{1}_{3Q}\mathbf{T}^{\alpha}\left(\mathbf{1}_{Q}\sigma\right)\right\|_{L^{2}(\omega)} \leq \mathfrak{T}_{\mathbf{T}^{\alpha}}^{\text{triple}}\left\|\mathbf{1}_{Q}\right\|_{L^{2}(\sigma)}, \qquad \text{for all quasicubes } Q \text{ in } \mathbb{R}^{n},$$

or the tripled dual quasicube testing condition defined with σ and ω interchanged and the dual operator $\mathbf{T}^{\alpha,*}$ in place of $\mathbf{T}^{\alpha}$. In turn, the tripled quasicube testing condition can be obtained from the quasicube testing condition for the truncated weight pairs $(\omega, \mathbf{1}_{Q}\sigma)$. See also Remark 4 below.

2.4. **Poisson integrals and $\mathcal{A}_2^{\alpha}$.** Recall that we have fixed a globally biLipschitz map $\Omega: \mathbb{R}^n \to \mathbb{R}^n$. Now let μ be a locally finite positive Borel measure on $\mathbb{R}^n$, and suppose Q is an Ω-quasicube in $\mathbb{R}^n$. Recall that $|Q|^{\frac{1}{n}} \approx \ell(Q)$ for a quasicube Q. The two α-fractional Poisson integrals of μ on a quasicube Q are given by:

$$\begin{aligned}
\mathrm{P}^{\alpha}(Q,\mu) &\equiv \int_{\mathbb{R}^n} \frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}}+|x-x_Q|\right)^{n+1-\alpha}} d\mu(x), \\
\mathcal{P}^{\alpha}(Q,\mu) &\equiv \int_{\mathbb{R}^n} \left(\frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}}+|x-x_Q|\right)^{2}}\right)^{n-\alpha} d\mu(x),
\end{aligned}$$

where we emphasize that $|x-x_Q|$ denotes Euclidean distance between x and x_Q and $|Q|$ denotes the Lebesgue measure of the quasicube Q. We refer to P^{α} as the *standard* Poisson integral and to $\mathcal{P}^{\alpha}$ as the *reproducing* Poisson integral.

We say that the pair (K, K') in $\mathcal{P}^n \times \mathcal{P}^n$ are *neighbours* if K and K' live in a common dyadic grid and both $K \subset 3K' \setminus K'$ and $K' \subset 3K \setminus K$, and we denote by $\mathcal{N}^n$ the set of pairs (K, K') in $\mathcal{P}^n \times \mathcal{P}^n$ that are neighbours. Let

$$\Omega\mathcal{N}^n = \left\{(\Omega K, \Omega K') : (K, K') \in \mathcal{N}^n\right\}$$

be the corresponding collection of neighbour pairs of quasicubes. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$, possibly having common point masses, and suppose $0 \leq \alpha < n$. Then we define the classical *offset A_2^{α} constants* by

$$A_2^{\alpha}(\sigma,\omega) \equiv \sup_{(Q,Q') \in \Omega\mathcal{N}^n} \frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q'|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}}. \tag{2.3}$$

Since the cubes in $\mathcal{P}^n$ are products of half open, half closed intervals $[a,b)$, the neighbouring quasicubes $(Q, Q') \in \Omega\mathcal{N}^n$ are disjoint, and the common point masses of σ and ω do not simultaneously appear in each factor.

We now define the *one-tailed* $\mathcal{A}_2^{\alpha}$ constant using $\mathcal{P}^{\alpha}$. The energy constants $\mathcal{E}_{\alpha}$ introduced in the next subsection will use the standard Poisson integral P^{α}.

Definition 4. *The one-sided constants $\mathcal{A}_2^{\alpha}$ and $\mathcal{A}_2^{\alpha,*}$ for the weight pair (σ,ω) are given by*

$$\begin{aligned}
\mathcal{A}_2^{\alpha} &\equiv \sup_{Q \in \Omega\mathcal{P}^n} \mathcal{P}^{\alpha}\left(Q, \mathbf{1}_{Q^c}\sigma\right) \frac{|Q|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}} < \infty, \\
\mathcal{A}_2^{\alpha,*} &\equiv \sup_{Q \in \Omega\mathcal{P}^n} \mathcal{P}^{\alpha}\left(Q, \mathbf{1}_{Q^c}\omega\right) \frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}} < \infty.
\end{aligned}$$

Note that these definitions are the analogues of the corresponding conditions with 'holes' introduced by Hytönen [Hyt] in dimension $n=1$ - the supports of the measures $\mathbf{1}_{Q^c}\sigma$ and $\mathbf{1}_Q\omega$ in the definition of $\mathcal{A}_2^\alpha$ are disjoint, and so the common point masses of σ and ω do not appear simultaneously in each factor. Note also that, unlike in [SaShUr5], where common point masses were not permitted, we can no longer assert the equivalence of $\mathcal{A}_2^\alpha$ with holes taken over *quasicubes* with $\mathcal{A}_2^\alpha$ with holes taken over *cubes*.

2.4.1. *Punctured A_2^α conditions.* As mentioned earlier, the classical A_2^α condition

$$A_2^\alpha(\sigma,\omega) \equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}$$

fails to be finite when the measures σ and ω have a common point mass - simply let Q in the sup above shrink to a common mass point. But there is a substitute that is quite similar in character that is motivated by the fact that for large quasicubes Q, the sup above is problematic only if just *one* of the measures is *mostly* a point mass when restricted to Q. The one-dimensional version of the condition we are about to describe arose in Conjecture 1.12 of Lacey [Lac2], and it was pointed out in [Hyt2] that its necessity on the line follows from the proof of Proposition 2.1 in [LaSaUr2]. We now extend this condition to higher dimensions, where its necessity is more subtle.

Given an at most countable set $\mathfrak{P}=\{p_k\}_{k=1}^\infty$ in $\mathbb{R}^n$, a quasicube $Q\in\Omega\mathcal{Q}^n$, and a positive locally finite Borel measure μ, define

$$\mu(Q,\mathfrak{P}) \equiv |Q|_\mu - \sup\{\mu(p_k) : p_k\in Q\cap\mathfrak{P}\},$$

where the supremum is actually achieved since $\sum_{p_k\in Q\cap\mathfrak{P}}\mu(p_k)<\infty$ as μ is locally finite. The quantity $\mu(Q,\mathfrak{P})$ is simply the $\widetilde{\mu}$ measure of Q where $\widetilde{\mu}$ is the measure μ with its largest point mass from $\mathfrak{P}$ in Q removed. Given a locally finite measure pair (σ,ω), let $\mathfrak{P}_{(\sigma,\omega)}=\{p_k\}_{k=1}^\infty$ be the at most countable set of common point masses of σ and ω. Then the weighted norm inequality (1.2) typically implies finiteness of the following *punctured* Muckenhoupt conditions:

$$\begin{aligned}
A_2^{\alpha,\mathrm{punct}}(\sigma,\omega) &\equiv \sup_{Q\in\Omega\mathcal{P}^n}\frac{\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}},\\
A_2^{\alpha,*,\mathrm{punct}}(\sigma,\omega) &\equiv \sup_{Q\in\Omega\mathcal{P}^n}\frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}}\frac{\sigma\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}.
\end{aligned}$$

Lemma 1. *Let $\mathbf{T}^\alpha$ be an α-fractional singular integral operator as above, and suppose that there is a positive constant C_0 such that*

$$\sqrt{A_2^\alpha(\sigma,\omega)} \le C_0\mathfrak{N}_{\mathbf{T}^\alpha}(\sigma,\omega),$$

for all pairs of positive locally finite measures ***having no common point masses****. Now let σ and ω be positive locally finite Borel measures on $\mathbb{R}^n$ and let $\mathfrak{P}_{(\sigma,\omega)}$ be the possibly nonempty set of common point masses. Then we have*

$$A_2^{\alpha,\mathrm{punct}}(\sigma,\omega)+A_2^{\alpha,*,\mathrm{punct}}(\sigma,\omega) \le 4C_0\mathfrak{N}_{\mathbf{T}^\alpha}^2(\sigma,\omega).$$

Proof. Fix a quasicube $Q\in\Omega\mathcal{P}^n$. Suppose first that $\mathfrak{P}_{(\sigma,\omega)}\cap Q=\{p_k\}_{k=1}^{2N}$ is finite. Choose $k_1\in\mathbb{N}_{2N}=\{1,2,...,2N\}$ so that

$$\sigma(p_{k_1}) = \max_{k\in\mathbb{N}_{2N}}\sigma(p_k).$$

Then choose $k_2\in\mathbb{N}_{2N}\setminus\{k_1\}$ such that

$$\omega(p_{k_2}) = \max_{k\in\mathbb{N}_{2N}\setminus\{k_1\}}\omega(p_k).$$

Repeat this procedure so that

$$\begin{aligned}
\sigma\left(p_{k_{2m+1}}\right) &= \max_{k\in\mathbb{N}_{2N}\setminus\{k_1,...k_{2m}\}}\sigma(p_k), \qquad k_{2m+1}\in\mathbb{N}_{2N}\setminus\{k_1,...k_{2m}\},\\
\omega\left(p_{k_{2m+2}}\right) &= \max_{k\in\mathbb{N}_{2N}\setminus\{k_1,...k_{2m+1}\}}\omega(p_k), \qquad k_{2m+2}\in\mathbb{N}_{2N}\setminus\{k_1,...k_{2m+1}\},
\end{aligned}$$

for each $m \leq N-1$. It is now clear that both

$$\begin{aligned}
\sum_{i=0}^{N-1} \sigma\left(p_{k_{2i+1}}\right) &\geq \frac{1}{2}\sum_{k=1}^{2N} \sigma\left(p_k\right) = \frac{1}{2}\sigma\left(Q \cap \mathfrak{P}_{(\sigma,\omega)}\right), \\
\sum_{i=0}^{N-1} \omega\left(p_{k_{2i+2}}\right) &\geq \frac{1}{2}\sum_{k=2}^{2N} \omega\left(p_k\right) = \frac{1}{2}\left[\omega\left(Q \cap \mathfrak{P}_{(\sigma,\omega)}\right) - \omega\left(p_1\right)\right].
\end{aligned}$$

Now define new measures $\widetilde{\sigma}$ and $\widetilde{\omega}$ by

$$\widetilde{\sigma} \equiv \mathbf{1}_Q \sigma - \sum_{i=0}^{N-1} \sigma\left(p_{k_{2i+2}}\right)\delta_{p_{k_{2i+2}}} \text{ and } \widetilde{\omega} = \mathbf{1}_Q \omega - \sum_{i=0}^{N-1} \omega\left(p_{k_{2i+1}}\right)\delta_{p_{k_{2i+1}}}$$

so that

$$\begin{aligned}
|Q|_{\widetilde{\sigma}} &= |Q|_\sigma - \sum_{i=0}^{N-1} \sigma\left(p_{k_{2i+2}}\right) = |Q|_\sigma - \sigma\left(Q \cap \mathfrak{P}_{(\sigma,\omega)}\right) + \sum_{i=0}^{N-1} \sigma\left(p_{k_{2i+1}}\right) \\
&\geq |Q|_\sigma - \sigma\left(Q \cap \mathfrak{P}_{(\sigma,\omega)}\right) + \frac{1}{2}\sigma\left(Q \cap \mathfrak{P}_{(\sigma,\omega)}\right) \geq \frac{1}{2}|Q|_\sigma
\end{aligned}$$

and

$$\begin{aligned}
|Q|_{\widetilde{\omega}} &= |Q|_\omega - \sum_{i=0}^{N-1} \omega\left(p_{k_{2i+1}}\right) = |Q|_\omega - \omega\left(Q \cap \mathfrak{P}_{(\sigma,\omega)}\right) + \sum_{i=0}^{N-1} \omega\left(p_{k_{2i+2}}\right) \\
&\geq |Q|_\omega - \omega\left(Q \cap \mathfrak{P}_{(\sigma,\omega)}\right) + \frac{1}{2}\left[\omega\left(Q \cap \mathfrak{P}_{(\sigma,\omega)}\right) - \omega\left(p_1\right)\right] \geq \frac{1}{2}\omega\left(Q, \mathfrak{P}_{(\sigma,\omega)}\right).
\end{aligned}$$

Now $\widetilde{\sigma}$ and $\widetilde{\omega}$ have no common point masses and $\mathfrak{N}_{\mathbf{T}^\alpha}(\sigma,\omega)$ is monotone in each measure separately, so we have

$$\begin{aligned}
\frac{\omega\left(Q, \mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}} &\leq 4\frac{|Q|_{\widetilde{\omega}}}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_{\widetilde{\sigma}}}{|Q|^{1-\frac{\alpha}{n}}} \\
&\leq 4A_2^\alpha\left(\widetilde{\sigma}, \widetilde{\omega}\right) \\
&\leq 4C_0 \mathfrak{N}_{\mathbf{T}^\alpha}^2\left(\widetilde{\sigma}, \widetilde{\omega}\right) \leq 4C_0 \mathfrak{N}_{\mathbf{T}^\alpha}^2(\sigma,\omega).
\end{aligned}$$

Now take the supremum over $Q \in \Omega\mathcal{Q}^n$ to conclude that $A_2^{\alpha,\text{punct}}(\sigma,\omega) \leq 4C_0\mathfrak{N}_{\mathbf{T}^\alpha}^2(\sigma,\omega)$ if the number of common point masses in Q is finite. A limiting argument proves the general case. The dual inequality $A_2^{\alpha,*,\text{punct}}(\sigma,\omega) \leq 4C_0\mathfrak{N}_{\mathbf{T}^\alpha}^2(\sigma,\omega)$ now follows upon interchanging the measures σ and ω. $\square$

Now we turn to the definition of a quasiHaar basis of $L^2(\mu)$.

2.5. **A weighted quasiHaar basis.** Recall we have fixed a globally biLipschitz map $\Omega : \mathbb{R}^n \to \mathbb{R}^n$. We will use a construction of a quasiHaar basis in $\mathbb{R}^n$ that is adapted to a measure μ (c.f. [NTV2] for the nonquasi case). Given a dyadic quasicube $Q \in \Omega\mathcal{D}$, where $\mathcal{D}$ is a dyadic grid from $\mathcal{P}^n$, let $\triangle_Q^\mu$ denote orthogonal projection onto the finite dimensional subspace $L_Q^2(\mu)$ of $L^2(\mu)$ that consists of linear combinations of the indicators of the children $\mathfrak{C}(Q)$ of Q that have μ-mean zero over Q:

$$L_Q^2(\mu) \equiv \left\{ f = \sum_{Q' \in \mathfrak{C}(Q)} a_{Q'} \mathbf{1}_{Q'} : a_{Q'} \in \mathbb{R}, \int_Q f d\mu = 0 \right\}.$$

Then we have the important telescoping property for dyadic quasicubes $Q_1 \subset Q_2$:

$$\mathbf{1}_{Q_0}(x)\left(\sum_{Q \in [Q_1,Q_2]} \triangle_Q^\mu f(x)\right) = \mathbf{1}_{Q_0}(x)\left(\mathbb{E}_{Q_0}^\mu f - \mathbb{E}_{Q_2}^\mu f\right), \qquad Q_0 \in \mathfrak{C}(Q_1), \ f \in L^2(\mu). \tag{2.4}$$

We will at times find it convenient to use a fixed orthonormal basis $\left\{h_Q^{\mu,a}\right\}_{a \in \Gamma_n}$ of $L_Q^2(\mu)$ where $\Gamma_n \equiv \{0,1\}^n \setminus \{\mathbf{1}\}$ is a convenient index set with $\mathbf{1} = (1,1,...,1)$. Then $\left\{h_Q^{\mu,a}\right\}_{a \in \Gamma_n \text{ and } Q \in \Omega\mathcal{D}}$ is an orthonormal

basis for $L^2(\mu)$, with the understanding that we add the constant function $\mathbf{1}$ if μ is a finite measure. In particular we have

$$\|f\|_{L^2(\mu)}^2=\sum_{Q\in\Omega\mathcal{D}}\left\|\triangle_Q^\mu f\right\|_{L^2(\mu)}^2=\sum_{Q\in\Omega\mathcal{D}}\sum_{a\in\Gamma_n}\left|\widehat{f}(Q)\right|^2,$$

where

$$\left|\widehat{f}(Q)\right|^2\equiv\sum_{a\in\Gamma_n}\left|\left\langle f,h_Q^{\mu,a}\right\rangle_\mu\right|^2,$$

and the measure is suppressed in the notation. Indeed, this follows from (2.4) and Lebesgue's differentiation theorem for quasicubes. We also record the following useful estimate. If I' is any of the 2^n $\Omega\mathcal{D}$-children of I, and $a\in\Gamma_n$, then

$$|\mathbb{E}_{I'}^\mu h_I^{\mu,a}|\leq\sqrt{\mathbb{E}_{I'}^\mu\left(h_I^{\mu,a}\right)^2}\leq\frac{1}{\sqrt{|I'|_\mu}}. \tag{2.5}$$

2.6. **The strong quasienergy conditions.** Given a dyadic quasicube $K\in\Omega\mathcal{D}$ and a positive measure μ we define the quasiHaar projection $\mathsf{P}_K^\mu\equiv\sum_{J\in\Omega\mathcal{D}:\ J\subset K}\triangle_J^\mu$ on K by

$$\mathsf{P}_K^\mu f=\sum_{J\in\Omega\mathcal{D}:\ J\subset K}\sum_{a\in\Gamma_n}\langle f,h_J^{\mu,a}\rangle_\mu h_J^{\mu,a}\text{ and }\|\mathsf{P}_K^\mu f\|_{L^2(\mu)}^2=\sum_{J\in\Omega\mathcal{D}:\ J\subset K}\sum_{a\in\Gamma_n}\left|\langle f,h_J^{\mu,a}\rangle_\mu\right|^2,$$

and where a quasiHaar basis $\{h_J^{\mu,a}\}_{a\in\Gamma_n\text{ and }J\in\Omega\mathcal{D}\Omega}$ adapted to the measure μ was defined in the subsubsection on a weighted quasiHaar basis above.

Now we define various notions for quasicubes which are inherited from the same notions for cubes. The main objective here is to use the familiar notation that one uses for cubes, but now extended to Ω-quasicubes. We have already introduced the notions of quasigrids $\Omega\mathcal{D}$, and center, sidelength and dyadic associated to quasicubes $Q\in\Omega\mathcal{D}$, as well as quasiHaar functions, and we will continue to extend to quasicubes the additional familiar notions related to cubes as we come across them. We begin with the notion of *deeply embedded*. Fix a quasigrid $\Omega\mathcal{D}$, $\mathbf{r}\in\mathbb{N}$ and $0<\varepsilon<1$. We say that a dyadic quasicube J is $(\mathbf{r},\varepsilon)$-*deeply embedded* in a (not necessarily dyadic) quasicube K, which we write as $J\Subset_{\mathbf{r},\varepsilon}K$, when $J\subset K$ and both

$$\begin{aligned}\ell(J)&\leq 2^{-\mathbf{r}}\ell(K),\\ \operatorname{qdist}(J,\partial K)&\geq\frac{1}{2}\ell(J)^\varepsilon\ell(K)^{1-\varepsilon},\end{aligned} \tag{2.6}$$

where we define the quasidistance $\operatorname{qdist}(E,F)$ between two sets E and F to be the Euclidean distance $\operatorname{dist}\left(\Omega^{-1}E,\Omega^{-1}F\right)$ between the preimages $\Omega^{-1}E$ and $\Omega^{-1}F$ of E and F under the map Ω, and where we recall that $\ell(J)\approx|J|^{\frac{1}{n}}$. For the most part we will consider $J\Subset_{\mathbf{r},\varepsilon}K$ when J and K belong to a common quasigrid $\Omega\mathcal{D}$, but an exception is made when defining the strong energy constants below.

Recall that in dimension $n=1$, and for $\alpha=0$, the energy condition constant was defined by

$$\mathcal{E}^2\equiv\sup_{I=\dot\cup I_r}\frac{1}{|I|_\sigma}\sum_{r=1}^\infty\left(\frac{\mathrm{P}^\alpha(I_r,\mathbf{1}_I\sigma)}{|I_r|}\right)^2\left\|\mathsf{P}_{I_r}^\omega\mathbf{x}\right\|_{L^2(\omega)}^2,$$

where I, I_r and J are intervals in the real line. The extension to higher dimensions we use here is that of 'strong quasienergy condition' below. Later on, in the proof of the theorem, we will break down this strong quasienergy condition into various smaller quasienergy conditions, which are then used in different ways in the proof.

We define a quasicube K (not necessarily in $\Omega\mathcal{D}$) to be an *alternate* $\Omega\mathcal{D}$-quasicube if it is a union of 2^n $\Omega\mathcal{D}$-quasicubes K' with side length $\ell(K')=\frac{1}{2}\ell(K)$ (such quasicubes were called shifted in [SaShUr5], but that terminology conflicts with the more familiar notion of shifted quasigrid). Thus for any $\Omega\mathcal{D}$-quasicube L there are exactly 2^n alternate $\Omega\mathcal{D}$-quasicubes of twice the side length that contain L, and one of them is of course the $\Omega\mathcal{D}$-parent of L. We denote the collection of alternate $\Omega\mathcal{D}$-quasicubes by $\mathcal{A}\Omega\mathcal{D}$.

The extension of the energy conditions to higher dimensions in [SaShUr5] used the collection

$$\mathcal{M}_{\mathbf{r},\varepsilon-\text{deep}}(K)\equiv\{\text{maximal }J\Subset_{\mathbf{r},\varepsilon}K\}$$

of *maximal* $(\mathbf{r},\varepsilon)$-deeply embedded dyadic subquasicubes of a quasicube K (a subquasicube J of K is a *dyadic* subquasicube of K if $J\in\Omega\mathcal{D}$ when $\Omega\mathcal{D}$ is a dyadic quasigrid containing K). This collection of dyadic

subquasicubes of K is of course a pairwise disjoint decomposition of K. We also defined there a refinement and extension of the collection $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$ for certain K and each $\ell \geq 1$. For an alternate quasicube $K \in \mathcal{A}\Omega\mathcal{D}$, define $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$ to consist of the *maximal* $\mathbf{r}$-deeply embedded $\Omega\mathcal{D}$-dyadic subquasicubes J of K. (In the special case that K itself belongs to $\Omega\mathcal{D}$, then these definitions coincide.) Then in [SaShUr5] for $\ell \geq 1$ we defined for $K \in \mathcal{A}\Omega\mathcal{D}$ the refinement

$$\begin{aligned}\mathcal{M}^{\ell}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K) \quad &\equiv \quad \left\{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(\pi^{\ell} K'\right) \text{ for some } K' \in \mathfrak{C}(K) :\right. \\ &\qquad\qquad \left. J \subset L \text{ for some } L \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)\right\},\end{aligned}$$

where $\mathfrak{C}(K)$ is the obvious extension to alternate quasicubes K of the set of $\Omega\mathcal{D}$-dyadic children of a dyadic quasicube. Thus $\mathcal{M}^{\ell}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$ is the union, over all quasichildren K' of K, of those quasicubes in $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(\pi^{\ell} K'\right)$ that happen to be contained in some $L \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$. Note that $\mathcal{M}^{0}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$ is in general different than $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$. We then define the *strong* quasienergy condition as follows.

Definition 5. *Let $0 \leq \alpha < n$ and fix parameters $(\mathbf{r},\varepsilon)$. Suppose σ and ω are positive Borel measures on $\mathbb{R}^n$ possibly with common point masses. Then the* strong *quasienergy constant $\mathcal{E}^{\mathrm{strong}}_{\alpha}$ is defined by*

$$\begin{aligned}\left(\mathcal{E}^{\mathrm{strong}}_{\alpha}\right)^2 \quad &\equiv \quad \sup_{I=\dot\cup I_r} \frac{1}{|I|_{\sigma}} \sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{\mathbf{r},\varepsilon-\mathrm{deep}}(I_r)} \left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\omega}_{J}\mathbf{x}\right\|^2_{L^2(\omega)} \\ &\qquad + \sup_{\Omega\mathcal{D}} \sup_{I \in \mathcal{A}\Omega\mathcal{D}} \sup_{\ell \geq 0} \frac{1}{|I|_{\sigma}} \sum_{J \in \mathcal{M}^{\ell}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(I)} \left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\omega}_{J}\mathbf{x}\right\|^2_{L^2(\omega)} .\end{aligned}$$

Similarly we have a dual version of $\mathcal{E}^{\mathrm{strong}}_{\alpha}$ denoted $\mathcal{E}^{\mathrm{strong},*}_{\alpha}$, and both depend on $\mathbf{r}$ and ε as well as on n and α. An important point in this definition is that the quasicube I in the second line is permitted to lie *outside* the quasigrid $\Omega\mathcal{D}$, but only as an alternate dyadic quasicube $I \in \mathcal{A}\Omega\mathcal{D}$. In the setting of quasicubes we continue to use the linear function $\mathbf{x}$ in the final factor $\left\|\mathsf{P}^{\omega}_{J}\mathbf{x}\right\|^2_{L^2(\omega)}$ of each line, and not the pushforward of $\mathbf{x}$ by Ω. The reason of course is that this condition is used to capture the first order information in the Taylor expansion of a singular kernel. There is a logically weaker form of the quasienergy conditions that we discuss after stating our main theorem, but these refined quasienergy conditions are more complicated to state, and have as yet found no application - the strong energy conditions above suffice for use when one measure is compactly supported on a $C^{1,\delta}$ curve as in [SaShUr8].

2.7. **Statement of the Theorem.** We can now state our main quasicube two weight theorem for general measures allowing common point masses. Recall that $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map, and that $\Omega\mathcal{P}^n$ denotes the collection of all quasicubes in $\mathbb{R}^n$ whose preimages under Ω are usual cubes with sides parallel to the coordinate axes. Denote by $\Omega\mathcal{D} \subset \Omega\mathcal{P}^n$ a dyadic quasigrid in $\mathbb{R}^n$. For the purpose of obtaining necessity of $\mathcal{A}^{\alpha}_2$ in the range $\frac{n}{2} \leq \alpha < n$, we adapt to the setting of quasicubes the notion of strong ellipticity from [SaShUr5].

Definition 6. *Fix a globally biLipschitz map Ω. Let $\mathbf{T}^{\alpha} = \left\{T^{\alpha}_j\right\}_{j=1}^{J}$ be a vector of Calderón-Zygmund operators with standard kernels $\left\{K^{\alpha}_j\right\}_{j=1}^{J}$. We say that $\mathbf{T}^{\alpha}$ is* strongly elliptic *with respect to Ω if for each $m \in \{1,-1\}^n$, there is a sequence of coefficients $\left\{\lambda^m_j\right\}_{j=1}^{J}$ such that*

$$\left|\sum_{j=1}^{J} \lambda^m_j K^{\alpha}_j(x, x+t\mathbf{u})\right| \geq c t^{\alpha-n}, \qquad t \in \mathbb{R}, \tag{2.7}$$

holds for all *unit vectors $\mathbf{u}$ in the quasi-n-ant ΩV_m where*

$$V_m = \left\{x \in \mathbb{R}^n : m_i x_i > 0 \text{ for } 1 \leq i \leq n\right\}, \qquad m \in \{1,-1\}^n .$$

Theorem 1. *Suppose that T^{α} is a standard α-fractional Calderón-Zygmund operator on $\mathbb{R}^n$, and that ω and σ are positive Borel measures on $\mathbb{R}^n$ (possibly having common point masses). Set $T^{\alpha}_{\sigma} f = T^{\alpha}(f\sigma)$ for any smooth truncation of T^{α}_{σ}. Let $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ be a globally biLipschitz map.*

(1) *Suppose $0 \leq \alpha < n$. Then the operator T_σ^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.*

$$\left\| T_\sigma^\alpha f \right\|_{L^2(\omega)} \leq \mathfrak{N}_{T_\sigma^\alpha} \left\| f \right\|_{L^2(\sigma)}, \tag{2.8}$$

uniformly in smooth truncations of T^α, and moreover

$$\mathfrak{N}_{T_\sigma^\alpha} \leq C_\alpha \left(\sqrt{\mathcal{A}_2^\alpha + \mathcal{A}_2^{\alpha,*} + A_2^{\alpha,\mathrm{punct}} + A_2^{\alpha,*,\mathrm{punct}}} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\mathrm{strong}} + \mathcal{E}_\alpha^{\mathrm{strong},*} + \mathcal{WBP}_{T^\alpha} \right),$$

provided that the two dual $\mathcal{A}_2^\alpha$ conditions and the two dual punctured Muckenhoupt conditions all hold, and the two dual quasitesting conditions for T^α hold, the quasiweak boundedness property for T^α holds for a sufficiently large constant C depending on the goodness parameter $\mathbf{r}$, and provided that the two dual strong quasienergy conditions hold uniformly over all dyadic quasigrids $\Omega\mathcal{D} \subset \Omega\mathcal{P}^n$, i.e. $\mathcal{E}_\alpha^{\mathrm{strong}} + \mathcal{E}_\alpha^{\mathrm{strong},*} < \infty$, and where the goodness parameters $\mathbf{r}$ and ε implicit in the definition of the collections $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$ and $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}^{\ell}(K)$ appearing in the strong energy conditions, are fixed sufficiently large and small respectively depending only on n and α.

(2) *Conversely, suppose $0 \leq \alpha < n$ and that $\mathbf{T}^\alpha = \left\{ T_j^\alpha \right\}_{j=1}^J$ is a vector of Calderón-Zygmund operators with standard kernels $\left\{ K_j^\alpha \right\}_{j=1}^J$. In the range $0 \leq \alpha < \frac{n}{2}$, we assume the* ellipticity *condition from (*[SaShUr5]*): there is $c > 0$ such that for* each *unit vector $\mathbf{u}$ there is j satisfying*

$$\left| K_j^\alpha (x, x + t\mathbf{u}) \right| \geq c t^{\alpha - n}, \qquad t \in \mathbb{R}. \tag{2.9}$$

For the range $\frac{n}{2} \leq \alpha < n$, we assume the strong ellipticity condition in Definition 6 above. Furthermore, assume that each operator T_j^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$,

$$\left\| \left(T_j^\alpha \right)_\sigma f \right\|_{L^2(\omega)} \leq \mathfrak{N}_{T_j^\alpha} \left\| f \right\|_{L^2(\sigma)}.$$

Then the fractional $\mathcal{A}_2^\alpha$ conditions (with 'holes') hold as well as the punctured Muckenhoupt conditions, and moreover,

$$\sqrt{\mathcal{A}_2^\alpha + \mathcal{A}_2^{\alpha,*} + A_2^{\alpha,\mathrm{punct}} + A_2^{\alpha,*,\mathrm{punct}}} \leq C \mathfrak{N}_{\mathbf{T}^\alpha}.$$

Problem 1. *Given any strongly elliptic vector $\mathbf{T}^\alpha$ of classical α-fractional Calderón-Zygmund operators, it is an open question whether or not the usual (quasi or not) energy conditions are necessary for boundedness of $\mathbf{T}^\alpha$. See* [SaShUr4] *for a failure of* energy reversal *in higher dimensions - such an energy reversal was used in dimension $n = 1$ to prove the necessity of the energy condition for the Hilbert transform, and also in* [SaShUr3] *and* [LaSaShUrWi] *for the Riesz transforms and Cauchy transforms respectively when one of the measures is supported on a line, and in* [SaShUr8] *when one of the measures is supported on a $C^{1,\delta}$ curve.*

Remark 3. *If Definition 6 holds for some $\mathbf{T}^\alpha$ and Ω, then Ω must be fairly tame, in particular the logarithmic spirals in Example 1 are ruled out! On the other hand, the vector of Riesz transforms $\mathbf{R}^{\alpha,n}$ is easily seen to be strongly elliptic with respect to Ω if Ω satisfies the following* sector separation property*. Given a hyperplane H and a perpendicular line L intersecting at point P, there exist spherical cones S_H and S_L intersecting only at the point $P' = \Omega(P)$, such that $H' \equiv \Omega H \subset S_H$ and $L' \equiv \Omega L \subset S_L$ and*

$$\begin{aligned} \mathrm{dist}(x, \partial S_H) &\approx |x|, \qquad x \in H, \\ \mathrm{dist}(x, \partial S_L) &\approx |x|, \qquad x \in L. \end{aligned}$$

Examples of globally biLipshcitz maps Ω that satisfy the sector separation property include finite compositions of maps of the form

$$\Omega(x_1, x') = (x_1, x' + \psi(x_1)), \qquad (x_1, x') \in \mathbb{R}^n,$$

where $\psi : \mathbb{R} \to \mathbb{R}^{n-1}$ is a Lipschitz map with sufficiently small Lipschitz constant.

Remark 4. *In* [LaWi]*, in the setting of usual (nonquasi) cubes and measures having no common point masses, M. Lacey and B. Wick use the NTV technique of surgery to show that the weak boundedness property for the Riesz transform vector $\mathbf{R}^{\alpha,n}$ is implied by the $\mathcal{A}_2^\alpha$ and testing conditions, and this has the consequence of eliminating the weak boundedness property as a condition. Their proof of this implication extends to the more general operators T^α and quasicubes considered here, and so the quasiweak boundedness property can be dropped from the statement of Theorem 1. In any event, the weak boundedness property is necessary for the norm inequality, and as such can be viewed as a weak close cousin of the testing conditions.*

3. Proof of Theorem 1

We now give the proof of Theorem 1 in the following sections. Sections 5, 7 and 10 are largely taken verbatim from the corresponding sections of [SaShUr5], but are included here since their omission here would hinder the readability of an already complicated argument.

3.1. Good quasicubes and quasienergy conditions.

Here we extend the notion of goodness to quasicubes and define various refinements of the strong quasienergy conditions appearing in the main theorem above. These refinements represent the 'weakest' energy side conditions that suffice for use in our proof. We begin with goodness.

Definition 7. *Let* $\mathbf{r}\in\mathbb{N}$ *and* $0<\varepsilon<1$. *Fix a quasigrid* $\Omega\mathcal{D}$. *A dyadic quasicube* J *is* $(\mathbf{r},\varepsilon)$-good, *or simply* good, *if for* every *dyadic superquasicube* I, *it is the case that* ***either*** J *has side length greater than* $2^{-\mathbf{r}}$ *times that of* I, ***or*** $J\Subset_{\mathbf{r},\varepsilon}I$ *is* $(\mathbf{r},\varepsilon)$*-deeply embedded in* I.

Note that this definition simply asserts that a dyadic quasicube $J=\Omega J'$ is $(\mathbf{r},\varepsilon)$-good if and only if the cube J' is $(\mathbf{r},\varepsilon)$-good in the usual sense. Finally, we say that J is $\mathbf{r}$-*nearby* in K when $J\subset K$ and

$$\ell(J)>2^{-\mathbf{r}}\ell(K).$$

The parameters $\mathbf{r},\varepsilon$ will be fixed sufficiently large and small respectively later in the proof, and we denote the set of such good dyadic quasicubes by $\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\mathrm{good}}$, or simply $\Omega\mathcal{D}_{\mathrm{good}}$ when the goodness parameters $(\mathbf{r},\varepsilon)$ are understood. Note that if $J'\in\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\mathrm{good}}$ and if $J'\subset K\in\Omega\mathcal{D}$, then **either** J' is $\mathbf{r}$-*nearby* in K **or** $J'\subset J\Subset_{\mathbf{r},\varepsilon}K$.

Throughout the proof, it will be convenient to also consider pairs of quasicubes J,K where J is $(\boldsymbol{\rho},\varepsilon)$-*deeply embedded* in K, written $J\Subset_{\boldsymbol{\rho},\varepsilon}K$ and meaning (2.6) holds with the same $\varepsilon>0$ but with $\boldsymbol{\rho}$ in place of $\mathbf{r}$; as well as pairs of quasicubes J,K where J is $\boldsymbol{\rho}$-*nearby* in K, $\ell(J)>2^{-\boldsymbol{\rho}}\ell(K)$, for a parameter $\boldsymbol{\rho}\gg\mathbf{r}$ that will be fixed later. We define the smaller 'good' quasiHaar projection $\mathsf{P}_K^{\mathrm{good},\omega}=\mathsf{P}_K^{(\mathbf{r},\varepsilon)-\mathrm{good},\omega}$ by

$$\mathsf{P}_K^{\mathrm{good},\mu}f\equiv\sum_{J\in\mathcal{G}(K)}\triangle_J^{\mu}f=\sum_{J\in\mathcal{G}(K)}\sum_{a\in\Gamma_n}\left\langle f,h_J^{\mu,a}\right\rangle_\mu h_J^{\mu,a},$$

where $\mathcal{G}(K)$ consists of the good subcubes of K:

$$\mathcal{G}(K)\equiv\left\{J\in\Omega\mathcal{D}_{\mathrm{good}}:J\subset K\right\},$$

and also the larger 'subgood' quasiHaar projection $\mathsf{P}_K^{\mathrm{subgood},\mu}$ by

$$\mathsf{P}_K^{\mathrm{subgood},\mu}f\equiv\sum_{J\in\mathcal{M}_{\mathrm{good}}(K)}\sum_{J'\subset J}\triangle_{J'}^{\mu}f=\sum_{J\in\mathcal{M}_{\mathrm{good}}(K)}\sum_{J'\subset J}\sum_{a\in\Gamma_n}\left\langle f,h_{J'}^{\mu,a}\right\rangle_\mu h_{J'}^{\mu,a},$$

where $\mathcal{M}_{\mathrm{good}}(K)$ consists of the *maximal* good subcubes of K. We thus have

$$\begin{aligned}\left\|\mathsf{P}_K^{\mathrm{good},\mu}\mathbf{x}\right\|_{L^2(\mu)}^2&\leq\left\|\mathsf{P}_K^{\mathrm{subgood},\mu}\mathbf{x}\right\|_{L^2(\mu)}^2\\&\leq\left\|\mathsf{P}_I^{\mu}\mathbf{x}\right\|_{L^2(\mu)}^2=\int_I\left|\mathbf{x}-\left(\frac{1}{|I|_\mu}\int_I\mathbf{x}d\mu\right)\right|^2d\mu(x),\qquad\mathbf{x}=(x_1,...,x_n),\end{aligned}$$

where $\mathsf{P}_I^{\mu}\mathbf{x}$ is the orthogonal projection of the identity function $\mathbf{x}:\mathbb{R}^n\to\mathbb{R}^n$ onto the vector-valued subspace of $\oplus_{k=1}^nL^2(\mu)$ consisting of functions supported in I with μ-mean value zero.

Recall that the extension of the energy conditions to higher dimensions in [SaShUr5] used the collection

$$\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)\equiv\left\{\text{maximal }J\Subset_{\mathbf{r},\varepsilon}K\right\}$$

of *maximal* $\mathbf{r}$-deeply embedded dyadic subquasicubes of a quasicube K (a subquasicube J of K is a *dyadic* subquasicube of K if $J\in\Omega\mathcal{D}$ when $\Omega\mathcal{D}$ is a dyadic quasigrid containing K). We let $J^*=\gamma J$ where $\gamma\geq2$. Then the following bounded overlap property holds.

Lemma 2.

$$\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)}\mathbf{1}_{J^*}\leq\beta\mathbf{1}_{\bigcup_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)}J^*}\tag{3.1}$$

holds for some positive constant β depending only on $n,\gamma,\mathbf{r}$ and ε. If in addition we have $\gamma\leq 2^{\mathbf{r}(1-\varepsilon)}$, then $\gamma J\subset K$ for all $J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$, and consequently

$$\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)}\mathbf{1}_{J^{*}}\leq\beta\mathbf{1}_{K}\ . \tag{3.2}$$

Proof. To prove (3.1), we first note that there are at most $2^{n\mathbf{r}+1}$ quasicubes J for which $\ell(J)>2^{-\mathbf{r}}\ell(K)$. On the other hand, the maximal $\mathbf{r}$-deeply embedded subquasicubes J of K also satisfy the comparability condition

$$\frac{1}{2}\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon}\leq\mathrm{qdist}(J,K^{c})\leq C_{n}\left(2^{\mathbf{r}}\ell(J)\right)^{\varepsilon}\ell(K)^{1-\varepsilon}.$$

Now with $0<\varepsilon<1$ and $\gamma\geq 2$ fixed, let $y\in K$. Then if $y\in\gamma J$, we have

$$\begin{aligned}\frac{1}{2}\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon} &\leq \mathrm{qdist}(J,K^{c})\leq\gamma\ell(J)+\mathrm{qdist}(\gamma J,K^{c})\\ &\leq \gamma\ell(J)+\mathrm{qdist}(y,K^{c}).\end{aligned}$$

Now assume that $\frac{\ell(J)}{\ell(K)}\leq\left(\frac{1}{4\gamma}\right)^{\frac{1}{1-\varepsilon}}$. Then we have $\gamma\ell(J)\leq\frac{1}{4}\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon}$ and so

$$\frac{1}{4}\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon}\leq\mathrm{qdist}(y,K^{c}).$$

But we also have

$$\mathrm{qdist}(y,K^{c})\leq\ell(J)+dist(J,K^{c})\leq\ell(J)+C_{n}2^{\mathbf{r}\varepsilon}\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon}\leq\left(\frac{1}{4\gamma}+C_{n}2^{\mathbf{r}\varepsilon}\right)\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon},$$

and so altogether, under the assumption that $\frac{\ell(J)}{\ell(K)}\leq\left(\frac{1}{4\gamma}\right)^{\frac{1}{1-\varepsilon}}$, we have

$$\begin{aligned}\frac{1}{\frac{1}{4\gamma}+C_{n}2^{\mathbf{r}\varepsilon}}\mathrm{qdist}(y,K^{c}) &\leq \ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon}\leq 2\,\mathrm{qdist}(y,K^{c}),\\ \text{i.e. }\left(\frac{1}{\frac{1}{4\gamma}+C_{n}2^{\mathbf{r}\varepsilon}}\frac{\mathrm{qdist}(y,K^{c})}{\ell(K)^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}} &\leq \ell(J)\leq\left(2\frac{\mathrm{qdist}(y,K^{c})}{\ell(K)^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}},\end{aligned}$$

which shows that the number of $J's$ satisfying $y\in\gamma J$ and $\frac{\ell(J)}{\ell(K)}\leq\left(\frac{1}{4\gamma}\right)^{\frac{1}{1-\varepsilon}}$ is at most $C_{n}'\gamma^{n}\frac{1}{\varepsilon}\log_{2}\left(\frac{1}{2\gamma}+2C_{n}2^{\mathbf{r}\varepsilon}\right)$. The number of $J's$ satisfying $y\in\gamma J$ and $\frac{\ell(J)}{\ell(K)}>\left(\frac{1}{4\gamma}\right)^{\frac{1}{1-\varepsilon}}$ is at most $C_{n}'\gamma^{n}\frac{1}{1-\varepsilon}\log_{2}(4\gamma)$. This proves (3.1) with

$$\begin{aligned}\beta &= 2^{n\mathbf{r}+1}+C_{n}'\gamma^{n}\frac{1}{\varepsilon}\log_{2}\left(\frac{1}{2\gamma}+2C_{n}2^{\mathbf{r}\varepsilon}\right)+C_{n}'\gamma^{n}\frac{1}{1-\varepsilon}\log_{2}(4\gamma)\\ &\leq 2^{n\mathbf{r}+1}+C_{n}''\frac{1}{\varepsilon}(1+\mathbf{r}\varepsilon)\gamma^{n}+C_{n}''\frac{1}{1-\varepsilon}\gamma^{n}(1+\log_{2}\gamma).\end{aligned}$$

In order to prove (3.2) it suffices, by (3.1), to prove $\gamma J\subset K$ for all $J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$. But $J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$ implies

$$\frac{1}{2}\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon}\leq\mathrm{qdist}(J,K^{c})=\mathrm{qdist}(c_{J},K^{c})+\frac{1}{2}\ell(J).$$

We wish to show $\gamma J\subset K$, which is equivalent to

$$\gamma\frac{1}{2}\ell(J)\leq\mathrm{qdist}(c_{J},K^{c})=\mathrm{qdist}(J,K^{c})-\frac{1}{2}\ell(J).$$

But we have

$$\mathrm{qdist}(J,K^{c})-\frac{1}{2}\ell(J)\geq\frac{1}{2}\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon}-\frac{1}{2}\ell(J),$$

and so it suffices to show that

$$\frac{1}{2}\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon}-\frac{1}{2}\ell(J)\geq\gamma\frac{1}{2}\ell(J),$$

which is equivalent to

$$\gamma+1\leq\ell(J)^{\varepsilon-1}\ell(K)^{1-\varepsilon}.$$

But the smallest that $\ell(J)^{\varepsilon-1}\ell(K)^{1-\varepsilon}$ can get for $J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$ is $2^{\mathbf{r}(1-\varepsilon)}$, and this completes the proof. □

Remark 5. *The parameter γ only plays a role in defining various refinements of the strong energy condition in our main theorem. We will use these refinements throughout our proof, both to highlight the nature of what precisely is required in different situations, and also to provide the 'weakest' energy conditions under which our main theorem holds, namely under the assumptions $\mathcal{E}_\alpha^{\mathrm{deep}},\mathcal{E}_\alpha^{\mathrm{refined}}<\infty$ with $2\leq\gamma\leq 2^{\mathbf{r}(1-\varepsilon)}$, and the corresponding dual conditions.*

Recall the collection $\mathcal{M}^\ell_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)$, which is the union, over all quasichildren K' of K, of those quasicubes in $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(\pi^\ell K')$ that happen to be contained in some $L\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)$. Since $J\in\mathcal{M}^\ell_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)$ implies $\gamma J\subset K$, we also have from (3.1) and (3.2) that

(3.3)
$$\sum_{J\in\mathcal{M}^{(\ell)}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)}\mathbf{1}_{J^*}\leq\beta\mathbf{1}_{\bigcup_{J\in\mathcal{M}^{(\ell)}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)}J^*}\ \text{and}\ \sum_{J\in\mathcal{M}^{(\ell)}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)}\mathbf{1}_{J^*\cap K}\leq\beta\mathbf{1}_K\ ,\qquad\text{for each }\ell\geq 0,$$

and if $\gamma\leq 2^{(1-\varepsilon)\mathbf{r}}$, then

$$\sum_{J\in\mathcal{M}^{(\ell)}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)}\mathbf{1}_{J^*}\leq\beta\mathbf{1}_K\ ,\qquad\text{for each }\ell\geq 0.\tag{3.4}$$

Of course $\mathcal{M}^1_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)\supset\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$, but $\mathcal{M}^\ell_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)$ is in general a finer subdecomposition of K the larger ℓ is, and may in fact be empty.

Now we proceed to recall the definition of various quasienergy conditions from [SaShUr5], but with the additional natural restriction that $\gamma\leq 2^{\mathbf{r}(1-\varepsilon)}$. The point of this restriction is that the bounded overlap conditions (3.2) and (3.4) can be used effectively later on.

Definition 8. *Suppose σ and ω are positive Borel measures on $\mathbb{R}^n$ (possibly with common point masses) and fix $\gamma\geq 2$. Then the deep quasienergy condition constant $\mathcal{E}_\alpha^{\mathrm{deep}}$ and the refined quasienergy condition constant $\mathcal{E}_\alpha^{\mathrm{refined}}$ are given by*

$$\begin{aligned}\left(\mathcal{E}_\alpha^{\mathrm{deep}}\right)^2 &\equiv \sup_{I=\dot\cup I_r}\frac{1}{|I|_\sigma}\sum_{r=1}^\infty\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I_r)}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\sigma\right)}{|J|^{\frac1n}}\right)^2\left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|^2_{L^2(\omega)},\\ \left(\mathcal{E}_\alpha^{\mathrm{refined}}\right)^2 &\equiv \sup_{\Omega\mathcal{D}}\sup_{I\in\mathcal{A}\Omega\mathcal{D}}\sup_{\ell\geq 0}\frac{1}{|I|_\sigma}\sum_{J\in\mathcal{M}^\ell_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(I)}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\sigma\right)}{|J|^{\frac1n}}\right)^2\left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|^2_{L^2(\omega)}\ ,\end{aligned}$$

where $\sup_{\Omega\mathcal{D}}\sup_I$ in the second line is taken over all quasigrids $\Omega\mathcal{D}$ and alternate quasicubes $I\in\mathcal{A}\Omega\mathcal{D}$, and $\sup_{I=\dot\cup I_r}$ in the first line is taken over

1. *all dyadic quasigrids $\Omega\mathcal{D}$,*
2. *all $\Omega\mathcal{D}$-dyadic quasicubes I,*
3. *and all subpartitions $\{I_r\}_{r=1}^{N\text{ or }\infty}$ of the quasicube I into $\Omega\mathcal{D}$-dyadic subquasicubes I_r.*

Note that in the refined quasienergy conditions there is no outer decomposition $I=\dot\cup I_r$. There are similar definitions for the dual (backward) quasienergy conditions that simply interchange σ and ω everywhere. These definitions of the quasienergy conditions depend on the choice of γ and the goodness parameters $\mathbf{r}$ and ε. Note that we can 'partially' plug the γ-hole in the Poisson integral $\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus\gamma J}\sigma\right)$ for both $\mathcal{E}_\alpha^{\mathrm{deep}}$ and $\mathcal{E}_\alpha^{\mathrm{refined}}$ using the offset A_2^α condition and the bounded overlap property (3.4). Indeed, define

(3.5)
$$\begin{aligned}\left(\mathcal{E}_\alpha^{\mathrm{deep\,partial}}\right)^2 &\equiv \sup_{I=\dot\cup I_r}\frac{1}{|I|_\sigma}\sum_{r=1}^\infty\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I_r)}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus J}\sigma\right)}{|J|^{\frac1n}}\right)^2\left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|^2_{L^2(\omega)}\ ,\\ \left(\mathcal{E}_\alpha^{\mathrm{refined\,partial}}\right)^2 &\equiv \sup_{\Omega\mathcal{D}}\sup_I\sup_{\ell\geq 0}\frac{1}{|I|_\sigma}\sum_{J\in\mathcal{M}^\ell_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(I)}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{I\setminus J}\sigma\right)}{|J|^{\frac1n}}\right)^2\left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|^2_{L^2(\omega)}\ .\end{aligned}$$

An easy calculation shows that

$$\gamma J\subset I_r \text{ for all } J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(I_r\right) \text{ provided } \gamma\leq c_n 2^{(1-\varepsilon)\mathbf{r}}. \tag{3.6}$$

Thus if $\gamma\leq 2^{(1-\varepsilon)\mathbf{r}}$, we have both

$$\begin{aligned}
&\left(\mathcal{E}_{\alpha}^{\mathrm{deep\,partial}}\right)^2 \\
\lesssim\ & \sup_{I=\dot{\cup}I_r}\frac{1}{\left|I\right|_{\sigma}}\sum_{r=1}^{\infty}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I_r)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I\setminus\gamma J}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\left\Vert \mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\Vert_{L^2(\omega)}^2 \\
&+\sup_{I=\dot{\cup}I_r}\frac{1}{\left|I\right|_{\sigma}}\sum_{r=1}^{\infty}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I_r)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\gamma J\setminus J}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\left\Vert \mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\Vert_{L^2(\omega)}^2 \\
\lesssim\ & \left(\mathcal{E}_{\alpha}^{\mathrm{deep}}\right)^2+\sup_{I=\dot{\cup}I_r}\frac{1}{\left|I\right|_{\sigma}}\sum_{r=1}^{\infty}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I_r)}\left(\frac{\left|\gamma J\setminus J\right|_{\sigma}}{\left|J\right|^{1+\frac{1}{n}-\frac{\alpha}{n}}}\right)^2\left|J\right|^{\frac{2}{n}}\left|J\right|_{\omega} \\
\lesssim\ & \left(\mathcal{E}_{\alpha}^{\mathrm{deep}}\right)^2+A_2^{\alpha}\sup_{I=\dot{\cup}I_r}\frac{1}{\left|I\right|_{\sigma}}\sum_{r=1}^{\infty}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I_r)}\left|\gamma J\right|_{\sigma}\lesssim\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}\right)^2+\beta A_2^{\alpha}\ ,
\end{aligned}\tag{3.7}$$

and similarly

$$\left(\mathcal{E}_{\alpha}^{\mathrm{refined\,partial}}\right)^2\lesssim\left(\mathcal{E}_{\alpha}^{\mathrm{refined}}\right)^2+\beta A_2^{\alpha}\ . \tag{3.8}$$

by (3.2) and (3.4) respectively.

Notation 2. *As above, we will typically use the side length $\ell(J)$ of a Ω-quasicube when we are describing collections of quasicubes, and when we want $\ell(J)$ to be a dyadic or related number; while in estimates we will typically use $\left|J\right|^{\frac{1}{n}}\approx\ell(J)$, and when we want to compare powers of volumes of quasicubes. We will continue to use the prefix 'quasi' when discussing quasicubes, quasiHaar, quasienergy and quasidistance in the text, but will not use the prefix 'quasi' when discussing other notions. In particular, since* quasi $\mathcal{A}_2^{\alpha}+$ quasi $A_2^{\alpha,\mathrm{punct}}\approx\mathcal{A}_2^{\alpha}+A_2^{\alpha,\mathrm{punct}}$ *(see e.g.* [SaShUr8] *for a proof) we do not use* quasi *as a prefix for the Muckenhoupt conditions, even though* quasi $\mathcal{A}_2^{\alpha}$ *alone is not comparable to* $\mathcal{A}_2^{\alpha}$*. Finally, we will not modify any mathematical symbols to reflect quasinotions, except for using $\Omega\mathcal{D}$ to denote a quasigrid, and* $\mathrm{qdist}(E,F)\equiv\mathrm{dist}\left(\Omega^{-1}E,\Omega^{-1}F\right)$ *to denote quasidistance between sets E and F, and using $\left|x-y\right|_{\mathrm{qdist}}\equiv\left|\Omega^{-1}x-\Omega^{-1}y\right|$ to denote quasidistance between points x and y. This limited use of quasi in the text serves mainly to remind the reader we are working entirely in the 'quasiworld'.*

3.1.1. *Plugged quasienergy conditions.* We now use the punctured conditions $A_2^{\alpha,\mathrm{punct}}$ and $A_2^{\alpha,*,\mathrm{punct}}$ to control the *plugged* quasienergy conditions, where the hole in the argument of the Poisson term $\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I\setminus J}\sigma\right)$ in the partially plugged quasienergy conditions above, is replaced with the 'plugged' term $\mathrm{P}^{\alpha}\left(J,\mathbf{1}_I\sigma\right)$. The resulting plugged quasienergy condition constants will be denoted by

$$\mathcal{E}_{\alpha}^{\mathrm{deep\,plug}},\mathcal{E}_{\alpha}^{\mathrm{refined\,plug}}\text{ and }\mathcal{E}_{\alpha}\equiv\mathcal{E}_{\alpha}^{\mathrm{deep}}+\mathcal{E}_{\alpha}^{\mathrm{refined}},$$

for example

$$\left(\mathcal{E}_{\alpha}^{\mathrm{deep\,plug}}\right)^2\equiv\sup_{I=\dot{\cup}I_r}\frac{1}{\left|I\right|_{\sigma}}\sum_{r=1}^{\infty}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I_r)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_I\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\left\Vert \mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\Vert_{L^2(\omega)}^2\ . \tag{3.9}$$

We first show that the punctured Muckenhoupt conditions $A_2^{\alpha,\mathrm{punct}}$ and $A_2^{\alpha,*,\mathrm{punct}}$ control respectively the 'energy A_2^{α} conditions', denoted $A_2^{\alpha,\mathrm{energy}}$ and $A_2^{\alpha,*,\mathrm{energy}}$ where

$$\begin{aligned}
A_2^{\alpha,\mathrm{energy}}(\sigma,\omega) &\equiv \sup_{Q\in\Omega\mathcal{P}^n}\frac{\left\Vert \mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell(Q)}\right\Vert_{L^2(\omega)}^2}{\left|Q\right|^{1-\frac{\alpha}{n}}}\frac{\left|Q\right|_{\sigma}}{\left|Q\right|^{1-\frac{\alpha}{n}}}, \\
A_2^{\alpha,*,\mathrm{energy}}(\sigma,\omega) &\equiv \sup_{Q\in\Omega\mathcal{P}^n}\frac{\left|Q\right|_{\omega}}{\left|Q\right|^{1-\frac{\alpha}{n}}}\frac{\left\Vert \mathsf{P}_Q^{\sigma}\frac{\mathbf{x}}{\ell(Q)}\right\Vert_{L^2(\sigma)}^2}{\left|Q\right|^{1-\frac{\alpha}{n}}}.
\end{aligned}\tag{3.10}$$

Lemma 3. *For any positive locally finite Borel measures σ,ω we have*

$$\begin{aligned} A_2^{\alpha,\mathrm{energy}}(\sigma,\omega) &\leq \max\{n,3\} A_2^{\alpha,\mathrm{punct}}(\sigma,\omega), \\ A_2^{\alpha,*,\mathrm{energy}}(\sigma,\omega) &\leq \max\{n,3\} A_2^{\alpha,*,\mathrm{punct}}(\sigma,\omega). \end{aligned}$$

Proof. Fix a quasicube $Q\in\Omega\mathcal{D}$. If $\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)\geq\frac{1}{2}|Q|_\omega$, then we trivially have

$$\begin{aligned} \frac{\left\|\mathsf{P}_Q^\omega\frac{\mathbf{x}}{\ell(Q)}\right\|_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}} &\leq n\frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}} \\ &\leq 2n\frac{\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\leq 2nA_2^{\alpha,\mathrm{punct}}(\sigma,\omega). \end{aligned}$$

On the other hand, if $\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)<\frac{1}{2}|Q|_\omega$ then there is a point $p\in Q\cap\mathfrak{P}_{(\sigma,\omega)}$ such that

$$\omega(\{p\})>\frac{1}{2}|Q|_\omega\ ,$$

and consequently, p is the largest ω-point mass in Q. Thus if we define $\widetilde{\omega}=\omega-\omega(\{p\})\delta_p$, then we have

$$\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)=|Q|_{\widetilde{\omega}}\ .$$

Now we observe from the construction of Haar projections that

$$\triangle_J^{\widetilde{\omega}}=\triangle_J^\omega,\qquad \text{for all } J\in\Omega\mathcal{D} \text{ with } p\notin J.$$

So for each $s\geq 0$ there is a unique quasicube $J_s\in\Omega\mathcal{D}$ with $\ell(J_s)=2^{-s}\ell(Q)$ that contains the point p. For this quasicube we have, if $\{h_J^{\omega,a}\}_{J\in\Omega\mathcal{D},\ a\in\Gamma_n}$ is a basis for $L^2(\omega)$,

$$\begin{aligned} \left\|\triangle_{J_s}^\omega\mathbf{x}\right\|_{L^2(\omega)}^2 &= \sum_{a\in\Gamma_n}\left|\left\langle h_{J_s}^{\omega,a},x\right\rangle_\omega\right|^2=\sum_{a\in\Gamma_n}\left|\left\langle h_{J_s}^{\omega,a},x-p\right\rangle_\omega\right|^2 \\ &= \sum_{a\in\Gamma_n}\left|\int_{J_s}h_{J_s}^{\omega,a}(x)(x-p)\,d\omega(x)\right|^2=\sum_{a\in\Gamma_n}\left|\int_{J_s}h_{J_s}^{\omega,a}(x)(x-p)\,d\widetilde{\omega}(x)\right|^2 \\ &\leq \sum_{a\in\Gamma_n}\left\|h_{J_s}^{\omega,a}\right\|_{L^2(\widetilde{\omega})}^2\left\|\mathbf{1}_{J_s}(x-p)\right\|_{L^2(\widetilde{\omega})}^2\leq\sum_{a\in\Gamma_n}\left\|h_{J_s}^{\omega,a}\right\|_{L^2(\omega)}^2\left\|\mathbf{1}_{J_s}(x-p)\right\|_{L^2(\widetilde{\omega})}^2 \\ &\leq n2^n\ell(J_s)^2|J_s|_{\widetilde{\omega}}\leq 2^{-2s}\ell(Q)^2|Q|_{\widetilde{\omega}}. \end{aligned}$$

Thus we can estimate

$$\begin{aligned} \left\|\mathsf{P}_Q^\omega\frac{\mathbf{x}}{\ell(Q)}\right\|_{L^2(\omega)}^2 &= \frac{1}{\ell(Q)^2}\sum_{J\in\Omega\mathcal{D}:\ J\subset Q}\left\|\triangle_J^\omega\mathbf{x}\right\|_{L^2(\omega)}^2 \\ &= \frac{1}{\ell(Q)^2}\left(\sum_{J\in\Omega\mathcal{D}:\ p\notin J\subset Q}\left\|\triangle_J^{\widetilde{\omega}}\mathbf{x}\right\|_{L^2(\widetilde{\omega})}^2+\sum_{s=0}^\infty\left\|\triangle_{J_s}^\omega\mathbf{x}\right\|_{L^2(\omega)}^2\right) \\ &\leq \frac{1}{\ell(Q)^2}\left(\left\|\mathsf{P}_Q^{\widetilde{\omega}}\mathbf{x}\right\|_{L^2(\widetilde{\omega})}^2+\sum_{s=0}^\infty 2^{-2s}\ell(Q)^2|Q|_{\widetilde{\omega}}\right) \\ &\leq \frac{1}{\ell(Q)^2}\left(\ell(Q)^2|Q|_{\widetilde{\omega}}+\sum_{s=0}^\infty 2^{-2s}\ell(Q)^2|Q|_{\widetilde{\omega}}\right) \\ &\leq 3|Q|_{\widetilde{\omega}}\leq 3\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right), \end{aligned}$$

and so

$$\frac{\left\|\mathsf{P}_Q^\omega\frac{\mathbf{x}}{\ell(Q)}\right\|_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\leq\frac{3\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\leq 3A_2^{\alpha,\mathrm{punct}}(\sigma,\omega).$$

Now take the supremum over $Q\in\Omega\mathcal{D}$ to obtain $A_2^{\alpha,\mathrm{energy}}(\sigma,\omega)\leq\max\{n,3\}A_2^{\alpha,\mathrm{punct}}(\sigma,\omega)$. The dual inequality follows upon interchanging the measures σ and ω. $\square$

The next corollary follows immediately from Lemma 3 and the argument used above in (3.7) with (3.6). Define

$$\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2=\left(\mathcal{E}_{\alpha}^{\mathrm{deep\,plug}}\right)^2+\left(\mathcal{E}_{\alpha}^{\mathrm{refined\,plug}}\right)^2 .$$

Corollary 1. *Provided* $\gamma \leq c_n 2^{(1-\varepsilon)\mathbf{r}}$,

$$\begin{aligned}
\mathcal{E}_{\alpha}^{\mathrm{deep\,plug}} &\lesssim \mathcal{E}_{\alpha}^{\mathrm{deep\,partial}}+A_2^{\alpha,\mathrm{punct}} \lesssim \mathcal{E}_{\alpha}^{\mathrm{deep}}+A_2^{\alpha}+A_2^{\alpha,\mathrm{punct}},\\
\mathcal{E}_{\alpha}^{\mathrm{refined\,plug}} &\lesssim \mathcal{E}_{\alpha}^{\mathrm{refined\,partial}}+A_2^{\alpha,\mathrm{punct}} \lesssim \mathcal{E}_{\alpha}^{\mathrm{refined}}+A_2^{\alpha}+A_2^{\alpha,\mathrm{punct}},\\
\mathcal{E}_{\alpha}^{\mathrm{plug}} &\lesssim \mathcal{E}_{\alpha}+A_2^{\alpha}+A_2^{\alpha,\mathrm{punct}} .
\end{aligned}$$

and similarly for the dual plugged quasienergy conditions.

The two constants $\mathcal{E}_{\alpha}^{\mathrm{deep\,plug}}$ and $\mathcal{E}_{\alpha}^{\mathrm{refined\,plug}}$, but with a larger projection P_J^{ω}, are what we bundled together as the strong energy constant $\mathcal{E}_{\alpha}^{\mathrm{strong}}$ in the statement of our main result, Theorem 1.

3.1.2. *Plugged* $\mathcal{A}_2^{\alpha,\mathrm{energy\,plug}}$ *conditions.* Using Lemma 3 we can control the 'plugged' energy $\mathcal{A}_2^{\alpha}$ conditions:

$$\begin{aligned}
\mathcal{A}_2^{\alpha,\mathrm{energy\,plug}}(\sigma,\omega) &\equiv \sup_{Q\in\Omega\mathcal{P}^n}\frac{\left\|\mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell(Q)}\right\|_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}(Q,\sigma),\\
\mathcal{A}_2^{\alpha,*,\mathrm{energy\,plug}}(\sigma,\omega) &\equiv \sup_{Q\in\Omega\mathcal{P}^n}\mathcal{P}^{\alpha}(Q,\omega)\frac{\left\|\mathsf{P}_Q^{\sigma}\frac{\mathbf{x}}{\ell(Q)}\right\|_{L^2(\sigma)}^2}{|Q|^{1-\frac{\alpha}{n}}}.
\end{aligned}$$

Lemma 4. *We have*

$$\begin{aligned}
\mathcal{A}_2^{\alpha,\mathrm{energy\,plug}}(\sigma,\omega) &\lesssim \mathcal{A}_2^{\alpha}(\sigma,\omega)+A_2^{\alpha,\mathrm{energy}}(\sigma,\omega),\\
\mathcal{A}_2^{\alpha,*,\mathrm{energy\,plug}}(\sigma,\omega) &\lesssim \mathcal{A}_2^{\alpha,*}(\sigma,\omega)+A_2^{\alpha,*,\mathrm{energy}}(\sigma,\omega).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\frac{\left\|\mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell(Q)}\right\|_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}(Q,\sigma) &= \frac{\left\|\mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell(Q)}\right\|_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}(Q,\mathbf{1}_{Q^c}\sigma)+\frac{\left\|\mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell(Q)}\right\|_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}(Q,\mathbf{1}_{Q}\sigma)\\
&\lesssim \frac{|Q|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}(Q,\mathbf{1}_{Q^c}\sigma)+\frac{\left\|\mathsf{P}_Q^{\omega}\frac{\mathbf{x}}{\ell(Q)}\right\|_{L^2(\omega)}^2}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}}\\
&\lesssim \mathcal{A}_2^{\alpha}(\sigma,\omega)+A_2^{\alpha,\mathrm{energy}}(\sigma,\omega).
\end{aligned}$$

□

3.2. **Random grids.** Using the analogue for dyadic quasigrids of the good random grids of Nazarov, Treil and Volberg, a standard argument of NTV, see e.g. [Vol], reduces the two weight inequality (1.2) for T^{α} to proving boundedness of a bilinear form $\mathcal{T}^{\alpha}(f,g)$ with uniform constants over dyadic quasigrids, and where the quasiHaar supports $\operatorname{supp}\widehat{f}$ and $\operatorname{supp}\widehat{g}$ of the functions f and g are contained in the collection $\Omega\mathcal{D}^{\mathrm{good}}$ of good quasicubes, whose children are all good as well, with goodness parameters $\mathbf{r}<\infty$ and $\varepsilon>0$ chosen sufficiently large and small respectively depending only on n and α. Here the quasiHaar support of f is $\operatorname{supp}\widehat{f} \equiv \{I\in\Omega\mathcal{D}:\triangle_I^{\sigma}f\neq 0\}$, and similarly for g. In fact we can assume even more, namely that the quasiHaar supports $\operatorname{supp}\widehat{f}$ and $\operatorname{supp}\widehat{g}$ of f and g are contained in the collection of $\boldsymbol{\tau}$*-good* quasicubes

$$\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\mathrm{good}}^{\boldsymbol{\tau}} \equiv \left\{K\in\Omega\mathcal{D}:\mathfrak{C}_K\subset\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\mathrm{good}} \text{ and } \pi_{\Omega\mathcal{D}}^{\ell}K\in\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\mathrm{good}} \text{ for all } 0\leq\ell\leq\boldsymbol{\tau}\right\}, \tag{3.11}$$

that are $(\mathbf{r},\varepsilon)$-good and whose children are also $(\mathbf{r},\varepsilon)$-good, and whose ℓ-parents up to level $\boldsymbol{\tau}$ are also $(\mathbf{r},\varepsilon)$-good. Here $\boldsymbol{\tau}>\mathbf{r}$ is a parameter to be fixed in Definition 12 below. We may assume this restriction on the quasiHaar supports of f and g by the following lemma.

Lemma 5. *Given* $\mathbf{r}\geq 3$, $\boldsymbol{\tau}\geq 1$ *and* $\frac{1}{\mathbf{r}}<\varepsilon<1-\frac{1}{\mathbf{r}}$, *we have*

$$\Omega\mathcal{D}_{(\mathbf{r}-1,\delta)-\mathrm{good}}\subset\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\mathrm{good}}^{\boldsymbol{\tau}} ,$$

provided

$$0<\delta\leq\frac{\mathbf{r}\varepsilon-1}{\mathbf{r}+\boldsymbol{\tau}}. \tag{3.12}$$

Proof. Suppose that $I\in\Omega\mathcal{D}_{(\mathbf{r}-1,\delta)-\mathrm{good}}$ where δ is as in (3.12). If J is a child of I, then $J\in\Omega\mathcal{D}_{(\mathbf{r},\delta)-\mathrm{good}}$, and since $\delta<\varepsilon$, we also have $\Omega\mathcal{D}_{(\mathbf{r},\delta)-\mathrm{good}}\subset\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\mathrm{good}}$. It remains to show that $\pi_{\Omega\mathcal{D}}^{(m)}I\in\Omega\mathcal{D}_{(\mathbf{r},\varepsilon)-\mathrm{good}}$ for $0\leq m\leq\boldsymbol{\tau}$. For this it suffices to show that if $K\in\Omega\mathcal{D}$ satisfies $\pi_{\Omega\mathcal{D}}^{(m)}I\subset K$ and $\ell\left(\pi_{\Omega\mathcal{D}}^{(m)}I\right)\leq 2^{-\mathbf{r}}\ell\left(K\right)$, then

$$\frac{1}{2}\left(\frac{\ell\left(\pi_{\Omega\mathcal{D}}^{(m)}I\right)}{\ell\left(K\right)}\right)^{\varepsilon}\ell\left(K\right)\leq\mathrm{qdist}\left(\pi_{\Omega\mathcal{D}}^{(m)}I,K^{c}\right). \tag{3.13}$$

Now $\ell\left(I\right)=2^{-m}\ell\left(\pi_{\Omega\mathcal{D}}^{(m)}I\right)\leq 2^{-(m+\mathbf{r})}\ell\left(K\right)\leq 2^{-(\mathbf{r}-1)}\ell\left(K\right)$ and $I\in\Omega\mathcal{D}_{(\mathbf{r}-1,\delta)-\mathrm{good}}$ imply that

$$\frac{1}{2}\left(\frac{\ell\left(I\right)}{\ell\left(K\right)}\right)^{\delta}\ell\left(K\right)\leq\mathrm{qdist}\left(I,K^{c}\right),$$

and since the triangle inequality gives

$$\mathrm{qdist}\left(I,K^{c}\right)\leq\mathrm{qdist}\left(\pi_{\Omega\mathcal{D}}^{(m)}I,K^{c}\right)+2^{m}\ell\left(I\right),$$

we see that it suffices to show

$$\frac{1}{2}\left(\frac{\ell\left(\pi_{\Omega\mathcal{D}}^{(m)}I\right)}{\ell\left(K\right)}\right)^{\varepsilon}\ell\left(K\right)+2^{m}\ell\left(I\right)\leq\frac{1}{2}\left(\frac{\ell\left(I\right)}{\ell\left(K\right)}\right)^{\delta}\ell\left(K\right),\qquad 0\leq m\leq\boldsymbol{\tau}. \tag{3.14}$$

This is equivalent to successively,

$$\begin{aligned}
\frac{1}{2}\left(\frac{2^{m}\ell\left(I\right)}{\ell\left(K\right)}\right)^{\varepsilon}\ell\left(K\right)+2^{m}\ell\left(I\right) &\leq \frac{1}{2}\left(\frac{\ell\left(I\right)}{\ell\left(K\right)}\right)^{\delta}\ell\left(K\right);\\
\left(\frac{2^{m}\ell\left(I\right)}{\ell\left(K\right)}\right)^{\varepsilon}+2^{m+1}\frac{\ell\left(I\right)}{\ell\left(K\right)} &\leq \left(\frac{\ell\left(I\right)}{\ell\left(K\right)}\right)^{\delta};\\
2^{m\varepsilon}\left(\frac{\ell\left(I\right)}{\ell\left(K\right)}\right)^{\varepsilon-\delta}+2^{m+1}\left(\frac{\ell\left(I\right)}{\ell\left(K\right)}\right)^{1-\delta} &\leq 1,\qquad 0\leq m\leq\boldsymbol{\tau}.
\end{aligned}$$

Since $0<\delta<\varepsilon<1$ by our restriction on ε and our choice of δ in (3.12), and since $\frac{\ell(I)}{\ell(K)}\leq 2^{-(m+\mathbf{r})}$, it thus suffices to show that

$$\begin{aligned}
2^{m\varepsilon}\left(2^{-(m+\mathbf{r})}\right)^{\varepsilon-\delta}+2^{m+1}\left(2^{-(m+\mathbf{r})}\right)^{1-\delta} &\leq 1,\\
\text{i.e. } 2^{m\varepsilon-(m+\mathbf{r})(\varepsilon-\delta)}+2^{m+1-(m+\mathbf{r})(1-\delta)} &\leq 1,
\end{aligned}$$

for $0\leq m\leq\boldsymbol{\tau}$. In particular then it suffices to show both

$$\begin{aligned}
m\varepsilon-\left(m+\mathbf{r}\right)\left(\varepsilon-\delta\right) &\leq -1,\\
m+1-\left(m+\mathbf{r}\right)\left(1-\delta\right) &\leq -1,
\end{aligned}$$

equivalently both

$$\begin{aligned}
\left(\mathbf{r}+m\right)\delta &\leq \mathbf{r}\varepsilon-1,\\
\left(\mathbf{r}+m\right)\delta &\leq \mathbf{r}-2,
\end{aligned}$$

for $0\leq m\leq\boldsymbol{\tau}$. Finally then it suffices to show both

$$\delta\leq\frac{\mathbf{r}\varepsilon-1}{\mathbf{r}+\boldsymbol{\tau}}\text{ and }\delta\leq\frac{\mathbf{r}-2}{\mathbf{r}+\boldsymbol{\tau}}.$$

Because of the restriction that $\frac{1}{\mathbf{r}}<\varepsilon<1-\frac{1}{\mathbf{r}}$, we see that $0<\mathbf{r}\varepsilon-1<\mathbf{r}-2$, and it is now clear that the above display holds for our choice of δ in (3.12). $\square$

For convenience in notation we will sometimes suppress the dependence on α in our nonlinear forms, but will retain it in the operators, Poisson integrals and constants. More precisely, let $\Omega\mathcal{D}^{\sigma}=\Omega\mathcal{D}^{\omega}$ be an $(\mathbf{r},\varepsilon)$-good quasigrid on $\mathbb{R}^n$, and let $\{h_I^{\sigma,a}\}_{I\in\Omega\mathcal{D}^{\sigma},\ a\in\Gamma_n}$ and $\left\{h_J^{\omega,b}\right\}_{J\in\Omega\mathcal{D}^{\omega},\ b\in\Gamma_n}$ be corresponding quasiHaar bases as described above, so that

$$\begin{aligned} f &= \sum_{I\in\Omega\mathcal{D}^{\sigma}}\triangle_I^{\sigma}f=\sum_{I\in\Omega\mathcal{D}^{\sigma},\ a\in\Gamma_n}\langle f,h_I^{\sigma,a}\rangle\ h_I^{\sigma,a}=\sum_{I\in\Omega\mathcal{D}^{\sigma},\ a\in\Gamma_n}\widehat{f}(I;a)\ h_I^{\sigma,a},\\ g &= \sum_{J\in\Omega\mathcal{D}^{\omega}}\triangle_J^{\omega}g=\sum_{J\in\Omega\mathcal{D}^{\omega},\ b\in\Gamma_n}\left\langle g,h_J^{\omega,b}\right\rangle\ h_J^{\omega,b}=\sum_{J\in\Omega\mathcal{D}^{\omega},\ b\in\Gamma_n}\widehat{g}(J;b)\ h_J^{\omega,b}, \end{aligned}$$

where the appropriate measure is understood in the notation $\widehat{f}(I;a)$ and $\widehat{g}(J;b)$, and where these quasiHaar coefficients $\widehat{f}(I;a)$ and $\widehat{g}(J;b)$ vanish if the quasicubes I and J are not good. Inequality (2.8) is equivalent to boundedness of the bilinear form

$$\mathcal{T}^{\alpha}(f,g)\equiv\langle T_{\sigma}^{\alpha}(f),g\rangle_{\omega}=\sum_{I\in\Omega\mathcal{D}^{\sigma}\text{ and }J\in\Omega\mathcal{D}^{\omega}}\langle T_{\sigma}^{\alpha}(\triangle_I^{\sigma}f),\triangle_J^{\omega}g\rangle_{\omega}$$

on $L^2(\sigma)\times L^2(\omega)$, i.e.

$$|\mathcal{T}^{\alpha}(f,g)|\leq\mathfrak{N}_{T^{\alpha}}\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)},$$

uniformly over all quasigrids and appropriate truncations. We may assume the two quasigrids $\Omega\mathcal{D}^{\sigma}$ and $\Omega\mathcal{D}^{\omega}$ are equal here, and this we will do throughout the paper, although we sometimes continue to use the measure as a superscript on $\Omega\mathcal{D}$ for clarity of exposition. Roughly speaking, we analyze the form $\mathcal{T}^{\alpha}(f,g)$ by splitting it in a nonlinear way into three main pieces, following in part the approach in [LaSaShUr2] and [LaSaShUr3]. The first piece consists of quasicubes I and J that are either disjoint or of comparable side length, and this piece is handled using the section on preliminaries of NTV type. The second piece consists of quasicubes I and J that overlap, but are 'far apart' in a nonlinear way, and this piece is handled using the sections on the Intertwining Proposition and the control of the functional quasienergy condition by the quasienergy condition. Finally, the remaining local piece where the overlapping quasicubes are 'close' is handled by generalizing methods of NTV as in [LaSaShUr], and then splitting the stopping form into two sublinear stopping forms, one of which is handled using techniques of [LaSaUr2], and the other using the stopping time and recursion of M. Lacey [Lac]. See the schematic diagram in Subsection 7.4 below.

We summarize our assumptions on the Haar supports of f and g, and on the dyadic quasigrids $\Omega\mathcal{D}$.

Condition 1 (on Haar supports and quasigrids)**.** *We suppose the quasiHaar supports of the functions f and g satisfy* $\operatorname{supp}\widehat{f},\operatorname{supp}\widehat{g}\subset\Omega\mathcal{D}^{\tau}_{(\mathbf{r},\varepsilon)-\mathrm{good}}$. *We also assume that $|\partial Q|_{\sigma+\omega}=0$ for all dyadic quasicubes Q in the grids $\Omega\mathcal{D}$ (since this property holds with probability* 1 *for random grids $\Omega\mathcal{D}$).*

4. Necessity of the $\mathcal{A}_2^{\alpha}$ conditions

Here we prove in particular the necessity of the fractional $\mathcal{A}_2^{\alpha}$ condition (with holes) when $0\leq\alpha<n$, for the boundedness from $L^2(\sigma)$ to $L^2(\omega)$ (where σ and ω may have common point masses) of the α-fractional Riesz vector transform $\mathbf{R}^{\alpha}$ defined by

$$\mathbf{R}^{\alpha}(f\sigma)(x)=\int_{\mathbb{R}^n}K_j^{\alpha}(x,y)f(y)\,d\sigma(y),\qquad K_j^{\alpha}(x,y)=\frac{x^j-y^j}{|x-y|^{n+1-\alpha}},$$

whose kernel $K_j^{\alpha}(x,y)$ satisfies (2.1) for $0\leq\alpha<n$. See [SaShUr5] for the case without holes.

Lemma 6. *Suppose $0\leq\alpha<n$. Let T^{α} be any collection of operators with α-standard fractional kernel satisfying the ellipticity condition (2.9), and in the case $\frac{n}{2}\leq\alpha<n$, we also assume the more restrictive condition (2.7). Then for $0\leq\alpha<n$ we have*

$$\sqrt{\mathcal{A}_2^{\alpha}}\lesssim\mathfrak{N}_{\alpha}(T^{\alpha}).$$

Remark 6. *Cancellation properties of T^{α} play no role in the proof below. Indeed the proof shows that $\mathcal{A}_2^{\alpha}$ is dominated by the best constant C in the restricted inequality*

$$\left\|\chi_E T^{\alpha}(f\sigma)\right\|_{L^{2,\infty}(\omega)} \leq C\left\|f\right\|_{L^2(\sigma)}, \qquad E=\mathbb{R}^n\setminus supp\ f.$$

Proof. First we give the proof for the case when T^{α} is the α-fractional Riesz transform $\mathbf{R}^{\alpha}$, whose kernel is $\mathbf{K}^{\alpha}(x,y)=\frac{x-y}{|x-y|^{n+1-\alpha}}$. Define the 2^n generalized n-ants $\mathcal{Q}_m$ for $m\in\{-1,1\}^n$, and their translates $\mathcal{Q}_m(w)$ for $w\in\mathbb{R}^n$ by

$$\begin{aligned}\mathcal{Q}_m &= \{(x_1,...,x_n): m_k x_k>0\},\\ \mathcal{Q}_m(w) &= \{z: z-w\in\mathcal{Q}_m\}, \qquad w\in\mathbb{R}^n.\end{aligned}$$

Fix $m\in\{-1,1\}^n$ and a quasicube I. For $a\in\mathbb{R}^n$ and $r>0$ let

$$\begin{aligned}s_I(x) &= \frac{\ell(I)}{\ell(I)+|x-\zeta_I|},\\ f_{a,r}(y) &= \mathbf{1}_{\mathcal{Q}_{-m}(a)\cap B(0,r)}(y)\, s_I(y)^{n-\alpha},\end{aligned}$$

where ζ_I is the center of the cube I. Now

$$\begin{aligned}\ell(I)|x-y| &\leq \ell(I)|x-\zeta_I|+\ell(I)|\zeta_I-y|\\ &\leq [\ell(I)+|x-\zeta_I|][\ell(I)+|\zeta_I-y|]\end{aligned}$$

implies

$$\frac{1}{|x-y|}\geq\frac{1}{\ell(I)}s_I(x)\,s_I(y), \qquad x,y\in\mathbb{R}^n.$$

Now the key observation is that with $L\zeta\equiv m\cdot\zeta$, we have

$$L(x-y)=m\cdot(x-y)\geq|x-y|, \qquad x\in\mathcal{Q}_m(y),$$

which yields

$$\begin{aligned}L(\mathbf{K}^{\alpha}(x,y)) &= \frac{L(x-y)}{|x-y|^{n+1-\alpha}}\\ &\geq \frac{1}{|x-y|^{n-\alpha}}\geq\ell(I)^{\alpha-n}s_I(x)^{n-\alpha}s_I(y)^{n-\alpha},\end{aligned}\tag{4.1}$$

provided $x\in\mathcal{Q}_m(y)$. Now we note that $x\in\mathcal{Q}_m(y)$ when $x\in\mathcal{Q}_m(a)$ and $y\in\mathcal{Q}_{-m}(a)$ to obtain that for $x\in\mathcal{Q}_m(a)$,

$$\begin{aligned}L(T^{\alpha}(f_{a,r}\sigma)(x)) &= \int_{\mathcal{Q}_{-m}(a)\cap B(0,r)}\frac{L(x-y)}{|x-y|^{n+1-\alpha}}s_I(y)^{n-\alpha}d\sigma(y)\\ &\geq \ell(I)^{\alpha-n}s_I(x)^{n-\alpha}\int_{\mathcal{Q}_{-m}(a)\cap B(0,r)}s_I(y)^{2n-2\alpha}d\sigma(y).\end{aligned}$$

Applying $|L\zeta|\leq\sqrt{n}|\zeta|$ and our assumed two weight inequality for the fractional Riesz transform, we see that for $r>0$ large,

$$\begin{aligned}&\ell(I)^{2\alpha-2n}\int_{\mathcal{Q}_m(a)}s_I(x)^{2n-2\alpha}\left(\int_{\mathcal{Q}_{-m}(a)\cap B(0,r)}s_I(y)^{2n-2\alpha}d\sigma(y)\right)^2 d\omega(x)\\ &\leq\|LT(\sigma f_{a,r})\|^2_{L^2(\omega)}\lesssim\mathfrak{N}_{\alpha}(\mathbf{R}^{\alpha})^2\|f_{a,r}\|^2_{L^2(\sigma)}=\mathfrak{N}_{\alpha}(\mathbf{R}^{\alpha})^2\int_{\mathcal{Q}_{-m}(a)\cap B(0,r)}s_I(y)^{2n-2\alpha}d\sigma(y).\end{aligned}$$

Rearranging the last inequality, we obtain

$$\ell(I)^{2\alpha-2n}\int_{\mathcal{Q}_m(a)}s_I(x)^{2n-2\alpha}d\omega(x)\int_{\mathcal{Q}_{-m}(a)\cap B(0,r)}s_I(y)^{2n-2\alpha}d\sigma(y)\lesssim\mathfrak{N}_{\alpha}(\mathbf{R}^{\alpha})^2,$$

and upon letting $r\to\infty$,

$$\int_{\mathcal{Q}_m(a)}\frac{\ell(I)^{n-\alpha}}{(\ell(I)+|x-\zeta_I|)^{2n-2\alpha}}d\omega(x)\int_{\mathcal{Q}_{-m}(a)}\frac{\ell(I)^{n-\alpha}}{(\ell(I)+|y-\zeta_I|)^{2n-2\alpha}}d\sigma(y)\lesssim\mathfrak{N}_{\alpha}(\mathbf{R}^{\alpha})^2.$$

Note that the ranges of integration above are pairs of opposing n-ants.

Fix a quasicube Q, which without loss of generality can be taken to be centered at the origin, $\zeta_Q = 0$. Then choose $a = (2\ell(Q), 2\ell(Q))$ and $I = Q$ so that we have

$$\begin{aligned} &\left(\int_{\mathcal{Q}_m(a)} \frac{\ell(Q)^{n-\alpha}}{(\ell(Q)+|x|)^{2n-2\alpha}} d\omega(x)\right)\left(\ell(Q)^{\alpha-n}\int_Q d\sigma\right) \\ \leq\ & C_\alpha \int_{\mathcal{Q}_m(a)} \frac{\ell(Q)^{n-\alpha}}{(\ell(Q)+|x|)^{2n-2\alpha}} d\omega(x) \int_{\mathcal{Q}_{-m}(a)} \frac{\ell(Q)^{n-\alpha}}{(\ell(Q)+|y|)^{2n-2\alpha}} d\sigma(y) \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha)^2 . \end{aligned}$$

Now fix $m = (1,1,...,1)$ and note that there is a fixed N (independent of $\ell(Q)$) and a fixed collection of rotations $\{\rho_k\}_{k=1}^N$, such that the rotates $\rho_k \mathcal{Q}_m(a)$, $1 \leq k \leq N$, of the n-ant $\mathcal{Q}_m(a)$ cover the complement of the ball $B(0, 4\sqrt{n}\ell(Q))$:

$$B\left(0, 4\sqrt{n}\ell(Q)\right)^c \subset \bigcup_{k=1}^N \rho_k \mathcal{Q}_m(a) .$$

Then we obtain, upon applying the same argument to these rotated pairs of n-ants,

$$\left(\int_{B\left(0,4\sqrt{n}\ell(Q)\right)^c} \frac{\ell(Q)^{n-\alpha}}{(\ell(Q)+|x|)^{2n-2\alpha}} d\omega(x)\right)\left(\ell(Q)^{\alpha-n}\int_Q d\sigma\right) \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha)^2 . \tag{4.2}$$

Now we assume for the moment the offset A_2^α condition

$$\ell(Q)^{2(\alpha-n)}\left(\int_{Q'} d\omega\right)\left(\int_Q d\sigma\right) \leq A_2^\alpha ,$$

where Q' and Q are neighbouring quasicubes, i.e. $(Q', Q) \in \Omega\mathcal{N}^n$. If we use this offset inequality with Q' ranging over $3Q \setminus Q$, and then use the separation of $B(0, 4\sqrt{n}\ell(Q)) \setminus 3Q$ and Q to obtain the inequality

$$\ell(Q)^{2(\alpha-n)}\left(\int_{B\left(0,4\sqrt{n}\ell(Q)\right)\setminus 3Q} d\omega\right)\left(\int_Q d\sigma\right) \lesssim A_2^\alpha \ ,$$

together with (4.2), we obtain

$$\left(\int_{\mathbb{R}^n\setminus Q} \frac{\ell(Q)^{n-\alpha}}{(\ell(Q)+|x|)^{2n-2\alpha}} d\omega(x)\right)^{\frac{1}{2}}\left(\ell(Q)^{\alpha-n}\int_Q d\sigma\right)^{\frac{1}{2}} \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha) + \sqrt{A_2^\alpha},$$

or

$$\ell(Q)^\alpha\left(\frac{1}{|Q|}\int_{\mathbb{R}^n\setminus Q} \frac{1}{\left(1+\frac{|x-\zeta_Q|}{\ell(Q)}\right)^{2n-2\alpha}} d\omega(x)\right)^{\frac{1}{2}}\left(\frac{1}{|Q|}\int_Q d\sigma\right)^{\frac{1}{2}} \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha) + \sqrt{A_2^\alpha}.$$

Clearly we can reverse the roles of the measures ω and σ and obtain

$$\sqrt{\mathcal{A}_2^{\alpha,*}} \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha) + \sqrt{A_2^\alpha}$$

for the kernels $\mathbf{K}^\alpha$, $0 \leq \alpha < n$.

More generally, to obtain the case when T^α is elliptic and the offset A_2^α condition holds, we note that the key estimate (4.1) above extends to the kernel $\sum_{j=1}^J \lambda_j^m K_j^\alpha$ of $\sum_{j=1}^J \lambda_j^m T_j^\alpha$ in (2.7) if the n-ants above are replaced by thin cones of sufficently small aperture, and there is in addition sufficient separation between opposing cones, which in turn may require a larger constant than $4\sqrt{n}$ in the choice of Q' above.

Finally, we turn to showing that the offset A_2^α condition is implied by the norm inequality, i.e.

$$\begin{aligned} \sqrt{A_2^\alpha} &\equiv \sup_{(Q',Q)\in\Omega\mathcal{N}^n} \ell(Q)^\alpha\left(\frac{1}{|Q'|}\int_{Q'} d\omega\right)^{\frac{1}{2}}\left(\frac{1}{|Q|}\int_Q d\sigma\right)^{\frac{1}{2}} \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha) ; \\ \text{i.e. } &\left(\int_{Q'} d\omega\right)\left(\int_Q d\sigma\right) \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha)^2 |Q|^{2-\frac{2\alpha}{n}} , \qquad (Q', Q) \in \Omega\mathcal{N}^n. \end{aligned}$$

In the range $0 \leq \alpha < \frac{n}{2}$ where we only assume (2.9), we adapt a corresponding argument from [LaSaUr1].

The 'one weight' argument on page 211 of Stein [Ste] yields the *asymmetric* two weight A_2^α condition

$$|Q'|_\omega |Q|_\sigma \leq C\mathfrak{N}_\alpha (\mathbf{R}^\alpha)^2 |Q|^{2\left(1-\frac{\alpha}{n}\right)}, \tag{4.3}$$

where Q and Q' are quasicubes of equal side length r and distance $C_0 r$ apart for some (fixed large) positive constant C_0 (for this argument we choose the unit vector $\mathbf{u}$ in (2.9) to point in the direction from Q to Q'). In the one weight case treated in [Ste] it is easy to obtain from this (even for a *single* direction $\mathbf{u}$) the usual (symmetric) A_2 condition. Here we will have to employ a different approach.

Now recall (see Sec 2 of [Saw] for the case of usual cubes, and the case of half open, half closed quasicubes here is no different) that given an open subset Φ of $\mathbb{R}^n$, we can choose $R \geq 3$ sufficiently large, depending only on the dimension, such that if $\left\{Q_j^k\right\}_j$ are the dyadic quasicubes maximal among those dyadic quasicubes Q satisfying $RQ \subset \Phi$, then the following properties hold:

$$\left\{\begin{array}{ll} \text{(disjoint cover)} & \Phi = \bigcup_j Q_j \text{ and } Q_j \cap Q_i = \emptyset \text{ if } i \neq j \\ \text{(Whitney condition)} & RQ_j \subset \Phi \text{ and } 3RQ_j \cap \Phi^c \neq \emptyset \text{ for all } j \\ \text{(finite overlap)} & \sum_j \chi_{3Q_j} \leq C\chi_\Phi \end{array}\right. . \tag{4.4}$$

So fix a pair of neighbouring quasicubes $(Q_0', Q_0) \in \Omega\mathcal{N}^n$, and let $\{\mathsf{Q}_i\}_i$ be a Whitney decomposition into quasicubes of the set $\Phi \equiv (Q_0' \times Q_0) \setminus \mathfrak{D}$ relative to the diagonal $\mathfrak{D}$ in $\mathbb{R}^n \times \mathbb{R}^n$. Of course, there are no common point masses of ω in Q_0' and σ in Q_0 since the quasicubes Q_0' and Q_0 are disjoint. Note that if $\mathsf{Q}_i = Q_i' \times Q_i$, then (4.3) can be written

$$|\mathsf{Q}_i|_{\omega\times\sigma} \leq C\mathfrak{N}_\alpha (\mathbf{R}^\alpha)^2 |\mathsf{Q}_i|^{1-\frac{\alpha}{n}}, \tag{4.5}$$

where $\omega \times \sigma$ denotes product measure on $\mathbb{R}^n \times \mathbb{R}^n$. We choose R sufficiently large in the Whitney decomposition (4.4), depending on C_0, such that (4.5) holds for all the Whitney quasicubes Q_i. We have $\sum_i |\mathsf{Q}_i| = |Q' \times Q| = |Q|^2$.

Moreover, if $\mathsf{R} = Q' \times Q$ is a rectangle in $\mathbb{R}^n \times \mathbb{R}^n$ (i.e. Q', Q are quasicubes in $\mathbb{R}^n$), and if $\mathsf{R} = \dot{\cup}_i \mathsf{R}_i$ is a finite disjoint union of rectangles R_α, then by additivity of the product measure $\omega \times \sigma$,

$$|\mathsf{R}|_{\omega\times\sigma} = \sum_i |\mathsf{R}_i|_{\omega\times\sigma}.$$

Let $\mathsf{Q}_0 = Q_0' \times Q_0$ and set

$$\Lambda \equiv \left\{\mathsf{Q} = Q' \times Q : \mathsf{Q} \subset \mathsf{Q}_0, \ell(Q) = \ell(Q') \approx C_0^{-1} \operatorname{qdist}(Q, Q') \text{ and (4.3) holds}\right\}.$$

Divide Q_0 into $2n \times 2n = 4n^2$ congruent subquasicubes $Q_0^1, ..., Q_0^{4^n}$ of side length $\frac{1}{2}$, and set aside those $Q_0^j \in \Lambda$ (those for which (4.3) holds) into a collection of stopping cubes Γ. Continue to divide the remaining $Q_0^j \in \Lambda$ of side length $\frac{1}{4}$, and again, set aside those $Q_0^{j,i} \in \Phi$ into Γ, and continue subdividing those that remain. We continue with such subdivisions for N generations so that all the cubes *not* set aside into Γ have side length 2^{-N} . The important property these cubes have is that they all lie within distance $Cr2^{-N}$ of the diagonal $\mathfrak{D} = \{(x,x) : (x,x) \in Q_0' \times Q_0\}$ in $\mathsf{Q}_0 = Q_0' \times Q_0$ since (4.3) holds for all pairs of cubes Q' and Q of equal side length r having distance approximately $C_0 r$ apart. Enumerate the cubes in Γ as $\{\mathsf{Q}_i\}_i$ and those remaining that are not in Γ as $\{\mathsf{P}_j\}_j$. Thus we have the pairwise disjoint decomposition

$$\mathsf{Q}_0 = \left(\bigcup_i \mathsf{Q}_i\right) \bigcup \left(\bigcup_j \mathsf{P}_j\right).$$

The countable additivity of the product measure $\omega \times \sigma$ shows that

$$|\mathsf{Q}_0|_{\omega\times\sigma} = \sum_i |\mathsf{Q}_i|_{\omega\times\sigma} + \sum_j |\mathsf{P}_j|_{\omega\times\sigma}.$$

Now we have

$$\sum_i |\mathsf{Q}_i|_{\omega\times\sigma} \lesssim \sum_i \mathfrak{N}_\alpha (\mathbf{R}^\alpha)^2 |\mathsf{Q}_i|^{1-\frac{\alpha}{n}},$$

and

$$\begin{aligned}\sum_i |\mathsf{Q}_i|^{1-\frac{\alpha}{n}} &= \sum_{k\in\mathbb{Z}:\ 2^k\leq\ell(Q_0)} \sum_{i:\ \ell(Q_i)=2^k} \left(2^{2nk}\right)^{1-\frac{\alpha}{n}} \\ &\approx \sum_{k\in\mathbb{Z}:\ 2^k\leq\ell(Q_0)} \left(\frac{2^k}{\ell(Q_0)}\right)^{-n} \left(2^{2nk}\right)^{1-\frac{\alpha}{n}} \quad \text{(Whitney)} \\ &= \ell(Q_0)^n \sum_{k\in\mathbb{Z}:\ 2^k\leq\ell(Q_0)} 2^{nk\left(-1+2-\frac{2\alpha}{n}\right)} \\ &\leq C_\alpha \ell(Q_0)^n \ell(Q_0)^{n\left(1-\frac{2\alpha}{n}\right)} = C_\alpha |Q_0\times Q_0|^{2-\frac{2\alpha}{n}} = C_\alpha |\mathsf{Q}_0|^{1-\frac{\alpha}{n}},\end{aligned}$$

provided $0\leq\alpha<\frac{n}{2}$. Using that the side length of $\mathsf{P}_j = P_j\times P'_j$ is 2^{-N} and $dist(\mathsf{P}_j,\mathfrak{D})\leq C_r 2^{-N}$, we have the following limit,

$$\sum_j |\mathsf{P}_j|_{\omega\times\sigma} = \left|\bigcup_j \mathsf{P}_j\right|_{\omega\times\sigma} \to 0 \text{ as } N\to\infty,$$

since $\bigcup_j \mathsf{P}_j$ shrinks to the empty set as $N\to\infty$, and since locally finite measures such as $\omega\times\sigma$ are regular in Euclidean space. This completes the proof that $\sqrt{A_2^\alpha}\lesssim\mathfrak{N}_\alpha(\mathbf{R}^\alpha)$ for the range $0\leq\alpha<\frac{n}{2}$.

Now we turn to proving $\sqrt{A_2^\alpha}\lesssim\mathfrak{N}_\alpha(\mathbf{R}^\alpha)$ for the range $\frac{n}{2}\leq\alpha<n$, where we assume the stronger ellipticity condition (2.7). So fix a pair of neighbouring quasicubes $(K',K)\in\Omega\mathcal{N}^n$, and assume that $\sigma+\omega$ doesn't charge the intersection $\overline{K'}\cap\overline{K}$ of the closures of K' and K. It will be convenient to add another dimension by replacing n by $n+1$ and working with the preimages $Q'=\Omega^{-1}K'$ and $Q=\Omega^{-1}K$ that are usual cubes, and with the corresponding pullbacks $\widetilde{\omega}=\mathcal{L}_1\times\Omega^*\omega$ and $\widetilde{\sigma}=\mathcal{L}_1\times\Omega^*\sigma$ of the measures ω and σ; here $\mathcal{L}_1$ is one-dimensional Lebesgue measure. We may also assume that

$$Q'=[-1,0)\times\prod_{i=1}^n Q_i, \qquad Q=[0,1)\times\prod_{i=1}^n Q_i.$$

where $Q_i=[a_i,b_i]$ for $1\leq i\leq n$ (since the other cases are handled in similar fashion). It is important to note that we are considering the intervals Q_i here to be closed, and we will track this difference as we proceed.

Choose $\theta_1\in[a_1,b_1]$ so that both

$$\left|[-1,0)\times[a_1,\theta_1]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\omega}}, \quad \left|[-1,0)\times[\theta_1,b_1]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\omega}} \geq \frac{1}{2}|Q'|_{\widetilde{\omega}}.$$

Now denote the two intervals $[a_1,\theta_1]$ and $[\theta_1,b_1]$ by $[a_1^*,b_1^*]$ and $[a_1^{**},b_1^{**}]$ where the order is chosen so that

$$\left|[0,1)\times[a_1^*,b_1^*]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\sigma}} \leq \left|[0,1)\times[a_1^{**},b_1^{**}]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\sigma}}.$$

Then we have both

$$\begin{aligned}\left|[-1,0)\times[a_1^*,b_1^*]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\omega}} &\geq \frac{1}{2}|Q|_{\widetilde{\omega}}, \\ \left|[0,1)\times[a_1^{**},b_1^{**}]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\sigma}} &\geq \frac{1}{2}|Q|_{\widetilde{\sigma}}.\end{aligned}$$

Now choose $\theta_2\in[a_2,b_2]$ so that both

$$\left|[-1,0)\times[a_1^*,b_1^*]\times[a_2,\theta_2]\times\prod_{i=3}^n Q_i\right|_{\widetilde{\omega}}, \quad \left|[-1,0)\times[a_1^*,b_1^*]\times[\theta_2,b_2]\times\prod_{i=3}^n Q_i\right|_{\widetilde{\omega}} \geq \frac{1}{4}|Q|_{\widetilde{\omega}},$$

and denote the two intervals $[a_2,\theta_2]$ and $[\theta_2,b_2]$ by $[a_2^*,b_2^*]$ and $[a_2^{**},b_2^{**}]$ where the order is chosen so that

$$[0,1)\times\left|[a_1^{**},b_1^{**}]\times[a_2^*,b_2^*]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\sigma}} \leq \left|[0,1)\times[a_1^{**},b_1^{**}]\times[a_2^{**},b_2^{**}]\times\prod_{i=2}^n Q_i\right|_{\widetilde{\sigma}}.$$

Then we have both

$$\begin{aligned}\left|[-1,0)\times[a_1^*,b_1^*]\times[a_2^*,b_2^*]\times\prod_{i=3}^n Q_i\right|_{\widetilde{\omega}} &\geq \frac{1}{4}|Q|_{\widetilde{\omega}}\ ,\\ \left|[0,1)\times[a_1^{**},b_1^{**}]\times[a_2^{**},b_2^{**}]\times\prod_{i=3}^n Q_i\right|_{\widetilde{\sigma}} &\geq \frac{1}{4}|Q|_{\widetilde{\sigma}}\ .\end{aligned}$$

Then we choose $\theta_3\in[a_3,b_3]$ so that both

$$\begin{aligned}\left|[-1,0)\times[a_1^*,b_1^*]\times[a_2^*,b_2^*]\times[a_3,\theta_3]\times\prod_{i=4}^n Q_i\right|_{\widetilde{\omega}} &\geq \frac{1}{8}|Q|_{\widetilde{\omega}}\,,\\ \left|[-1,0)\times[a_1^*,b_1^*]\times[a_2^*,b_2^*]\times[\theta_3,b_3]\times\prod_{i=4}^n Q_i\right|_{\widetilde{\omega}} &\geq \frac{1}{8}|Q|_{\widetilde{\omega}}\,,\end{aligned}$$

and continuing in this way we end up with two rectangles,

$$\begin{aligned}G &\equiv [-1,0)\times[a_1^*,b_1^*]\times[a_2^*,b_2^*]\times...[a_n^*,b_n^*]\,,\\ H &\equiv [0,1)\times[a_1^{**},b_1^{**}]\times[a_2^{**},b_2^{**}]\times...[a_n^{**},b_n^{**}]\,,\end{aligned}$$

that satisfy

$$\begin{aligned}|G|_{\widetilde{\omega}} &= |[-1,0)\times[a_1^*,b_1^*]\times[a_2^*,b_2^*]\times...[a_n^*,b_n^*]|_{\widetilde{\omega}}\geq\frac{1}{2^n}|Q|_{\widetilde{\omega}}\,,\\ |H|_{\widetilde{\sigma}} &= |[0,1)\times[a_1^{**},b_1^{**}]\times[a_2^{**},b_2^{**}]\times...[a_n^{**},b_n^{**}]|_{\widetilde{\sigma}}\geq\frac{1}{2^n}|Q|_{\widetilde{\sigma}}\,.\end{aligned}$$

However, the quasirectangles ΩG and ΩH lie in opposing quasi-n-ants at the vertex $\Omega\theta=\Omega(\theta_1,\theta_2,...,\theta_n)$, and so we can apply (2.7) to obtain that for $x\in\Omega G$,

$$\begin{aligned}\left|\sum_{j=1}^J\lambda_j^m T_j^\alpha(\mathbf{1}_{\Omega H}\sigma)(x)\right| &= \left|\int_{\Omega H}\sum_{j=1}^J\lambda_j^m K_j^\alpha(x,y)\,d\sigma(y)\right|\\ &\gtrsim \int_{\Omega H}|x-y|^{\alpha-n}d\sigma(y)\gtrsim|\Omega Q|^{\frac{\alpha}{n}-1}|\Omega H|_\sigma\,.\end{aligned}$$

For the inequality above, we need to know that the distinguished point $\Omega\theta$ is not a common point mass of σ and ω, but this follows from our assumption that $\sigma+\omega$ doesn't charge the intersection $\overline{K'}\cap\overline{K}$ of the closures of K' and K. Then from the norm inequality we get

$$\begin{aligned}|\Omega G|_\omega\left(|\Omega Q|^{\frac{\alpha}{n}-1}|\Omega H|_\sigma\right)^2 &\lesssim \int_G\left|\sum_{j=1}^J\lambda_j^m T_j^\alpha(\mathbf{1}_{\Omega H}\sigma)\right|^2 d\omega\\ &\lesssim \mathfrak{N}^2_{\sum_{j=1}^J\lambda_j^m T_j^\alpha}\int\mathbf{1}_{\Omega H}^2 d\sigma=\mathfrak{N}^2_{\sum_{j=1}^J\lambda_j^m T_j^\alpha}|\Omega H|_\sigma\,,\end{aligned}$$

from which we deduce that

$$\begin{aligned}|\Omega Q|^{2\left(\frac{\alpha}{n}-1\right)}|\Omega Q'|_\omega|\Omega Q|_\sigma &\lesssim 2^{2n}|\Omega Q|^{2\left(\frac{\alpha}{n}-1\right)}|\Omega G|_\omega|\Omega H|_\sigma\lesssim 2^{2n}\mathfrak{N}^2_{\sum_{j=1}^J\lambda_j^m T_j^\alpha};\\ |K|^{2\left(\frac{\alpha}{n}-1\right)}|K'|_\omega|K|_\sigma &\lesssim 2^{2n}\mathfrak{N}^2_{\sum_{j=1}^J\lambda_j^m T_j^\alpha}\ ,\end{aligned}$$

and hence

$$A_2^\alpha\lesssim 2^{2n}\mathfrak{N}^2_{\sum_{j=1}^J\lambda_j^m T_j^\alpha}\ .$$

Thus we have obtained the offset A_2^α condition for pairs $(K',K)\in\Omega\mathcal{N}^n$ such that $\sigma+\omega$ doesn't charge the intersection $\overline{K'}\cap\overline{K}$ of the closures of K' and K. From this and the argument at the beginning of this proof, we obtain the one-tailed $\mathcal{A}_2^\alpha$ conditions. Indeed, we note that $|\partial(rQ)|_{\sigma+\omega}>0$ for only a countable number of dilates $r>1$, and so a limiting argument applies. This completes the proof of Lemma 6. □

5. Monotonicity Lemma and Energy lemma

The Monotonicity Lemma below will be used to prove the Energy Lemma, which is then used in several places in the proof of Theorem 1. The formulation of the Monotonicity Lemma with $m=2$ for cubes is due to M. Lacey and B. Wick [LaWi], and corrects that used in early versions of our paper [SaShUr5].

5.1. The Monotonicity Lemma.

For $0\leq\alpha<n$ and $m\in\mathbb{R}_+$, we recall the m-weighted fractional Poisson integral

$$\mathrm{P}_m^\alpha(J,\mu)\equiv\int_{\mathbb{R}^n}\frac{|J|^{\frac{m}{n}}}{\left(|J|^{\frac{1}{n}}+|y-c_J|\right)^{n+m-\alpha}}d\mu(y),$$

where $\mathrm{P}_1^\alpha(J,\mu)=\mathrm{P}^\alpha(J,\mu)$ is the standard Poisson integral.

Lemma 7 (Monotonicity)**.** *Suppose that I and J are quasicubes in $\mathbb{R}^n$ such that $J\subset 2J\subset I$, and that μ is a signed measure on $\mathbb{R}^n$ supported outside I. Finally suppose that T^α is a standard fractional singular integral on $\mathbb{R}^n$ with $0<\alpha<n$. Then we have the estimate*

$$\left\Vert\triangle_J^\omega T^\alpha\mu\right\Vert_{L^2(\omega)}\lesssim\Phi^\alpha(J,|\mu|),\tag{5.1}$$

where for a positive measure ν,

$$\begin{aligned}\Phi^\alpha(J,\nu)^2&\equiv\left(\frac{\mathrm{P}^\alpha(J,\nu)}{|J|^{\frac{1}{n}}}\right)^2\left\Vert\triangle_J^\omega\mathbf{x}\right\Vert_{L^2(\omega)}^2+\left(\frac{\mathrm{P}_{1+\delta}^\alpha(J,\nu)}{|J|^{\frac{1}{n}}}\right)^2\left\Vert\mathbf{x}-\mathbf{m}_J\right\Vert_{L^2(\mathbf{1}_J\omega)}^2\ ,\\ \mathbf{m}_J&\equiv\mathbb{E}_J^\omega\mathbf{x}=\frac{1}{|J|_\omega}\int_J\mathbf{x}d\omega.\end{aligned}$$

Proof. Let $\{h_J^{\omega,a}\}_{a\in\Gamma}$ be an orthonormal basis of $L_J^2(\mu)$ as in a previous subsection. Now we use the smoothness estimate (2.1), together with Taylor's formula and the vanishing mean of the quasiHaar functions $h_J^{\omega,a}$ and $\mathbf{m}_J\equiv\frac{1}{|J|_\mu}\int_J\mathbf{x}d\mu(x)\in J$, to obtain

$$\begin{aligned}\left|\langle T^\alpha\mu,h_J^{\omega,a}\rangle_\omega\right|&=\left|\int\left\{\int K^\alpha(x,y)\,h_J^{\omega,a}(x)\,d\omega(x)\right\}d\mu(y)\right|=\left|\int\left\langle K_y^\alpha,h_J^{\omega,a}\right\rangle_\omega d\mu(y)\right|\\ &=\left|\int\left\langle K_y^\alpha(x)-K_y^\alpha(\mathbf{m}_J),h_J^{\omega,a}\right\rangle_\omega d\mu(y)\right|\\ &\leq\left|\left\langle\left[\int\nabla K_y^\alpha(\mathbf{m}_J)\,d\mu(y)\right](\mathbf{x}-\mathbf{m}_J),h_J^{\omega,a}\right\rangle_\omega\right|\\ &\quad+\left\langle\left[\int\sup_{\boldsymbol{\theta}_J\in J}\left|\nabla K_y^\alpha(\boldsymbol{\theta}_J)-\nabla K_y^\alpha(\mathbf{m}_J)\right|d\mu(y)\right]\left|\mathbf{x}-\mathbf{m}_J\right|,\left|h_J^{\omega,a}\right|\right\rangle_\omega\\ &\lesssim C_{CZ}\frac{\mathrm{P}^\alpha(J,|\mu|)}{|J|^{\frac{1}{n}}}\left\Vert\triangle_J^\omega\mathbf{x}\right\Vert_{L^2(\omega)}+C_{CZ}\frac{\mathrm{P}_{1+\delta}^\alpha(J,|\mu|)}{|J|^{\frac{1}{n}}}\left\Vert\mathbf{x}-\mathbf{m}_J\right\Vert_{L^2(\mathbf{1}_J\omega)}.\end{aligned}$$

□

5.2. The Energy Lemma.

Suppose now we are given a subset $\mathcal{H}$ of the dyadic quasigrid $\Omega\mathcal{D}^\omega$. Let $\mathsf{P}_{\mathcal{H}}^\omega=\sum_{J\in\mathcal{H}}\triangle_J^\omega$ be the corresponding ω-quasiHaar projection. We define $\mathcal{H}^*\equiv\bigcup_{J\in\mathcal{H}}\{J'\in\Omega\mathcal{D}^\omega:J'\subset J\}$.

Lemma 8 (**Energy Lemma**)**.** *Let J be a quasicube in $\Omega\mathcal{D}^\omega$. Let Ψ_J be an $L^2(\omega)$ function supported in J and with ω-integral zero, and denote its quasiHaar support by $\mathcal{H}=\operatorname{supp}\widehat{\Psi_J}$. Let ν be a positive measure supported in $\mathbb{R}^n\setminus\gamma J$ with $\gamma\geq 2$, and for each $J'\in\mathcal{H}$, let $d\nu_{J'}=\varphi_{J'}d\nu$ with $|\varphi_{J'}|\leq 1$. Let T^α be a standard*

α-fractional singular integral operator with $0 \leq \alpha < n$. Then with $\delta' = \frac{\delta}{2}$ we have

$$\begin{aligned}
\left| \sum_{J' \in \mathcal{H}} \left\langle T^\alpha \left(\nu_{J'} \right), \triangle^\omega_{J'} \Psi_J \right\rangle_\omega \right| &\lesssim \left\| \Psi_J \right\|_{L^2(\omega)} \left(\frac{\mathrm{P}^\alpha \left(J, \nu \right)}{\left| J \right|^{\frac{1}{n}}} \right) \left\| \mathsf{P}^\omega_{\mathcal{H}} \mathbf{x} \right\|_{L^2(\omega)} \\
&\quad + \left\| \Psi_J \right\|_{L^2(\omega)} \frac{1}{\gamma^{\delta'}} \left(\frac{\mathrm{P}^\alpha_{1+\delta'} \left(J, \nu \right)}{\left| J \right|^{\frac{1}{n}}} \right) \left\| \mathsf{P}^\omega_{\mathcal{H}^*} \mathbf{x} \right\|_{L^2(\omega)} \\
&\lesssim \left\| \Psi_J \right\|_{L^2(\omega)} \left(\frac{\mathrm{P}^\alpha \left(J, \nu \right)}{\left| J \right|^{\frac{1}{n}}} \right) \left\| \mathsf{P}^\omega_{\mathcal{H}^*} \mathbf{x} \right\|_{L^2(\omega)} ,
\end{aligned}$$

and in particular the 'pivotal' bound

$$\left| \left\langle T^\alpha \left(\nu \right), \Psi_J \right\rangle_\omega \right| \leq C \left\| \Psi_J \right\|_{L^2(\omega)} \mathrm{P}^\alpha \left(J, \left| \nu \right| \right) \sqrt{\left| J \right|_\omega} .$$

Remark 7. *The first term on the right side of the energy inequality above is the 'big' Poisson integral P^α times the 'small' energy term $\left\| \mathsf{P}^\omega_{\mathcal{H}} \mathbf{x} \right\|^2_{L^2(\omega)}$ that is additive in $\mathcal{H}$, while the second term on the right is the 'small' Poisson integral $\mathrm{P}^\alpha_{1+\delta'}$ times the 'big' energy term $\left\| \mathsf{P}^\omega_{\mathcal{H}^*} \mathbf{x} \right\|_{L^2(\omega)}$ that is no longer additive in $\mathcal{H}$. The first term presents no problems in subsequent analysis due solely to the additivity of the 'small' energy term. It is the second term that must be handled by special methods. For example, in the Intertwining Proposition below, the interaction of the singular integral occurs with a pair of quasicubes $J \subset I$ at* highly separated *levels, where the goodness of J can exploit the decay δ' in the kernel of the 'small' Poisson integral $\mathrm{P}^\alpha_{1+\delta'}$ relative to the 'big' Poisson integral P^α, and results in a bound directly by the quasienergy condition. On the other hand, in the local recursion of M. Lacey at the end of the paper, the separation of levels in the pairs $J \subset I$ can be as* little *as a fixed parameter $\boldsymbol{\rho}$, and here we must first separate the stopping form into two sublinear forms that involve the two estimates respectively. The form corresponding to the smaller Poisson integral $\mathrm{P}^\alpha_{1+\delta'}$ is again handled using goodness and the decay δ' in the kernel, while the form corresponding to the larger Poisson integral P^α requires the stopping time and recursion argument of M. Lacey.*

Proof. Using the Monotonicity Lemma 7, followed by $\left| \nu_{J'} \right| \leq \nu$ and the Poisson equivalence

$$\frac{\mathrm{P}^\alpha_m \left(J', \nu \right)}{\left| J' \right|^{\frac{m}{n}}} \approx \frac{\mathrm{P}^\alpha_m \left(J, \nu \right)}{\left| J \right|^{\frac{m}{n}}}, \qquad J' \subset J \subset 2J, \quad \operatorname{supp} \nu \cap 2J = \emptyset, \tag{5.2}$$

we have

$$\begin{aligned}
&\left| \sum_{J' \in \mathcal{H}} \left\langle T^\alpha \left(\nu_{J'} \right), \triangle^\omega_{J'} \Psi_J \right\rangle_\omega \right| = \left| \sum_{J' \in \mathcal{H}} \left\langle \triangle^\omega_{J'} T^\alpha \left(\nu_{J'} \right), \triangle^\omega_{J'} \Psi_J \right\rangle_\omega \right| \\
\lesssim &\sum_{J' \in \mathcal{H}} \Phi^\alpha \left(J', \left| \nu_{J'} \right| \right) \left\| \triangle^\omega_{J'} \Psi_J \right\|_{L^2(\omega)} \\
\lesssim &\left(\sum_{J' \in \mathcal{H}} \left(\frac{\mathrm{P}^\alpha \left(J', \nu \right)}{\left| J' \right|^{\frac{1}{n}}} \right)^2 \left\| \triangle^\omega_{J'} \mathbf{x} \right\|^2_{L^2(\omega)} \right)^{\frac{1}{2}} \left(\sum_{J' \in \mathcal{H}} \left\| \triangle^\omega_{J'} \Psi_J \right\|^2_{L^2(\omega)} \right)^{\frac{1}{2}} \\
&+ \left(\sum_{J' \in \mathcal{H}} \left(\frac{\mathrm{P}^\alpha_{1+\delta} \left(J', \nu \right)}{\left| J' \right|^{\frac{1}{n}}} \right)^2 \sum_{J'' \subset J'} \left\| \triangle^\omega_{J''} \mathbf{x} \right\|^2_{L^2(\omega)} \right)^{\frac{1}{2}} \left(\sum_{J' \in \mathcal{H}} \left\| \triangle^\omega_{J'} \Psi_J \right\|^2_{L^2(\omega)} \right)^{\frac{1}{2}} \\
\lesssim &\left(\frac{\mathrm{P}^\alpha \left(J, \nu \right)}{\left| J \right|^{\frac{1}{n}}} \right) \left\| \mathsf{P}^\omega_{\mathcal{H}} \mathbf{x} \right\|_{L^2(\omega)} \left\| \Psi_J \right\|_{L^2(\omega)} + \frac{1}{\gamma^{\delta'}} \left(\frac{\mathrm{P}^\alpha_{1+\delta'} \left(J, \nu \right)}{\left| J \right|^{\frac{1}{n}}} \right) \left\| \mathsf{P}^\omega_{\mathcal{H}^*} \mathbf{x} \right\|_{L^2(\omega)} \left\| \Psi_J \right\|_{L^2(\omega)} .
\end{aligned}$$

The last inequality follows from

$$\begin{aligned}
&\sum_{J'\in\mathcal{H}}\left(\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J',\nu\right)}{\left|J'\right|^{\frac{1}{n}}}\right)^{2}\sum_{J''\subset J'}\left\Vert \bigtriangleup_{J''}^{\omega}\mathbf{x}\right\Vert _{L^{2}(\omega)}^{2}\\
=&\sum_{J''\subset J}\left\{ \sum_{J':\ J''\subset J'\subset J}\left(\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J',\nu\right)}{\left|J'\right|^{\frac{1}{n}}}\right)^{2}\right\} \left\Vert \bigtriangleup_{J''}^{\omega}\mathbf{x}\right\Vert _{L^{2}(\omega)}^{2}\\
\lesssim&\frac{1}{\gamma^{2\delta'}}\sum_{J''\in\mathcal{H}^{*}}\left(\frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J'',\nu\right)}{\left|J''\right|^{\frac{1}{n}}}\right)^{2}\left\Vert \bigtriangleup_{J''}^{\omega}\mathbf{x}\right\Vert _{L^{2}(\omega)}^{2}\ ,
\end{aligned}$$

which in turn follows from (recalling $\delta=2\delta'$ and using $\left|J'\right|^{\frac{1}{n}}+\left|y-c_{J'}\right|\approx\left|J\right|^{\frac{1}{n}}+\left|y-c_{J}\right|$ and $\frac{\left|J\right|^{\frac{1}{n}}}{\left|J\right|^{\frac{1}{n}}+\left|y-c_{J}\right|}\leq\frac{1}{\gamma}$ for $y\in\mathbb{R}^{n}\setminus\gamma J$)

$$\begin{aligned}
&\sum_{J':\ J''\subset J'\subset J}\left(\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J',\nu\right)}{\left|J'\right|^{\frac{1}{n}}}\right)^{2}\\
=&\sum_{J':\ J''\subset J'\subset J}\left|J'\right|^{\frac{2\delta}{n}}\left(\int_{\mathbb{R}^{n}\setminus\gamma J}\frac{1}{\left(\left|J'\right|^{\frac{1}{n}}+\left|y-c_{J'}\right|\right)^{n+1+\delta-\alpha}}d\nu\left(y\right)\right)^{2}\\
\lesssim&\sum_{J':\ J''\subset J'\subset J}\frac{1}{\gamma^{2\delta'}}\frac{\left|J'\right|^{\frac{2\delta}{n}}}{\left|J\right|^{\frac{2\delta}{n}}}\left(\int_{\mathbb{R}^{n}\setminus\gamma J}\frac{\left|J\right|^{\frac{\delta'}{n}}}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_{J}\right|\right)^{n+1+\delta'-\alpha}}d\nu\left(y\right)\right)^{2}\\
=&\frac{1}{\gamma^{2\delta'}}\left(\sum_{J':\ J''\subset J'\subset J}\frac{\left|J'\right|^{\frac{2\delta}{n}}}{\left|J\right|^{\frac{2\delta}{n}}}\right)\left(\frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J,\nu\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\lesssim\frac{1}{\gamma^{2\delta'}}\left(\frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J,\nu\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}.
\end{aligned}$$

Finally we have the 'pivotal' bound from (5.2) and

$$\sum_{J''\subset J}\left\Vert \bigtriangleup_{J''}^{\omega}\mathbf{x}\right\Vert _{L^{2}(\omega)}^{2}=\left\Vert \mathbf{x}-\mathbf{m}_{J}\right\Vert _{L^{2}(\mathbf{1}_{J}\omega)}^{2}\leq\left|J\right|^{\frac{2}{n}}\left|J\right|_{\omega}\ .$$

□

6. Preliminaries of NTV type

An important reduction of our theorem is delivered by the following two lemmas, that in the case of one dimension are due to Nazarov, Treil and Volberg (see [NTV4] and [Vol]). The proofs given there do not extend in standard ways to higher dimensions with common point masses, and we use the quasiweak boundedness property to handle the case of touching quasicubes, and an application of Schur's Lemma to handle the case of separated quasicubes. The first lemma below is Lemmas 8.1 and 8.7 in [LaWi] but with the larger constant $\mathcal{A}_{2}^{\alpha}$ there in place of the smaller constant A_{2}^{α} here. We emphasize that only the offset A_{2}^{α} condition is needed with testing and weak boundedness in these preliminary estimates.

Lemma 9. *Suppose T^{α} is a standard fractional singular integral with $0\leq\alpha<n$, and that all of the quasicubes $I\in\Omega\mathcal{D}^{\sigma}$, $J\in\Omega\mathcal{D}^{\omega}$ below are good with goodness parameters ε and $\mathbf{r}$. Fix a positive integer $\boldsymbol{\rho}>\mathbf{r}$. For $f\in L^{2}\left(\sigma\right)$ and $g\in L^{2}\left(\omega\right)$ we have*

$$\sum_{\substack{(I,J)\in\Omega\mathcal{D}^{\sigma}\times\Omega\mathcal{D}^{\omega}\\ 2^{-\boldsymbol{\rho}}\ell(I)\leq\ell(J)\leq2^{\boldsymbol{\rho}}\ell(I)}}\left|\left\langle T_{\sigma}^{\alpha}\left(\bigtriangleup_{I}^{\sigma}f\right),\bigtriangleup_{J}^{\omega}g\right\rangle _{\omega}\right|\lesssim\left(\mathfrak{T}_{\alpha}+\mathfrak{T}_{\alpha}^{*}+\mathcal{WBP}_{T^{\alpha}}+\sqrt{A_{2}^{\alpha}}\right)\left\Vert f\right\Vert _{L^{2}(\sigma)}\left\Vert g\right\Vert _{L^{2}(\omega)}\tag{6.1}$$

and

$$\sum_{\substack{(I,J)\in\Omega\mathcal{D}^{\sigma}\times\Omega\mathcal{D}^{\omega}\\ I\cap J=\emptyset\text{ and }\frac{\ell(J)}{\ell(I)}\notin\left[2^{-\boldsymbol{\rho}},2^{\boldsymbol{\rho}}\right]}}\left|\left\langle T_{\sigma}^{\alpha}\left(\bigtriangleup_{I}^{\sigma}f\right),\bigtriangleup_{J}^{\omega}g\right\rangle _{\omega}\right|\lesssim\sqrt{A_{2}^{\alpha}}\left\Vert f\right\Vert _{L^{2}(\sigma)}\left\Vert g\right\Vert _{L^{2}(\omega)}.\tag{6.2}$$

Lemma 10. *Suppose T^{α} is a standard fractional singular integral with $0 \leq \alpha < n$, that all of the quasicubes $I \in \Omega\mathcal{D}^{\sigma}, J \in \Omega\mathcal{D}^{\omega}$ below are good, that $\boldsymbol{\rho} > \mathbf{r}$, that $f \in L^2(\sigma)$ and $g \in L^2(\omega)$, that $\mathcal{F} \subset \Omega\mathcal{D}^{\sigma}$ and $\mathcal{G} \subset \Omega\mathcal{D}^{\omega}$ are σ-Carleson and ω-Carleson collections respectively, i.e.,*

$$\sum_{F' \in \mathcal{F}:\ F' \subset F} |F'|_{\sigma} \lesssim |F|_{\sigma}, \qquad F \in \mathcal{F}, \text{ and } \sum_{G' \in \mathcal{G}:\ G' \subset G} |G'|_{\omega} \lesssim |G|_{\omega}, \qquad G \in \mathcal{G},$$

that there are numerical sequences $\{\alpha_{\mathcal{F}}(F)\}_{F \in \mathcal{F}}$ and $\{\beta_{\mathcal{G}}(G)\}_{G \in \mathcal{G}}$ such that

$$\sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F)^2 |F|_{\sigma} \leq \|f\|^2_{L^2(\sigma)} \text{ and } \sum_{G \in \mathcal{G}} \beta_{\mathcal{G}}(G)^2 |G|_{\sigma} \leq \|g\|^2_{L^2(\sigma)}, \tag{6.3}$$

and finally that for each pair of quasicubes $(I, J) \in \Omega\mathcal{D}^{\sigma} \times \Omega\mathcal{D}^{\omega}$, there are bounded functions $\beta_{I,J}$ and $\gamma_{I,J}$ supported in $I \setminus 2J$ and $J \setminus 2I$ respectively, satisfying

$$\left\|\beta_{I,J}\right\|_{\infty}, \left\|\gamma_{I,J}\right\|_{\infty} \leq 1.$$

Then

$$\begin{aligned} &\sum_{\substack{(F,J) \in \mathcal{F} \times \Omega\mathcal{D}^{\omega} \\ F \cap J = \emptyset \text{ and } \ell(J) \leq 2^{-\boldsymbol{\rho}}\ell(F)}} \left|\left\langle T^{\alpha}_{\sigma}\left(\beta_{F,J}\mathbf{1}_F \alpha_{\mathcal{F}}(F)\right), \triangle^{\omega}_J g\right\rangle_{\omega}\right| \\ &+ \sum_{\substack{(I,G) \in \Omega\mathcal{D}^{\sigma} \times \mathcal{G} \\ I \cap G = \emptyset \text{ and } \ell(I) \leq 2^{-\boldsymbol{\rho}}\ell(G)}} \left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_I f\right), \gamma_{I,G}\mathbf{1}_G \beta_{\mathcal{G}}(G)\right\rangle_{\omega}\right| \\ \lesssim\ & \sqrt{\mathfrak{A}^{\alpha}_2}\, \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}. \end{aligned} \tag{6.4}$$

Remark 8. *If $\mathcal{F}$ and $\mathcal{G}$ are σ-Carleson and ω-Carleson collections respectively, and if $\alpha_{\mathcal{F}}(F) = \mathbb{E}^{\sigma}_F |f|$ and $\beta_{\mathcal{G}}(G) = \mathbb{E}^{\omega}_G |g|$, then the 'quasi' orthogonality condition (6.3) holds (here 'quasi' has a different meaning than quasi), and this special case of Lemma 10 serves as a basic example.*

Remark 9. *Lemmas 9 and 10 differ mainly in that an orthogonal collection of quasiHaar projections is replaced by a 'quasi' orthogonal collection of indicators $\{\mathbf{1}_F \alpha_{\mathcal{F}}(F)\}_{F \in \mathcal{F}}$. More precisely, the main difference between (6.2) and (6.4) is that a quasiHaar projection $\triangle^{\sigma}_I f$ or $\triangle^{\omega}_J g$ has been replaced with a constant multiple of an indicator $\mathbf{1}_F \alpha_{\mathcal{F}}(F)$ or $\mathbf{1}_G \beta_{\mathcal{G}}(G)$, and in addition, a bounded function is permitted to multiply the indicator of the quasicube having larger sidelength.*

Proof. Note that in (6.1) we have used the parameter $\boldsymbol{\rho}$ in the exponent rather than $\mathbf{r}$, and this is possible because the arguments we use here only require that there are finitely many levels of scale separating I and J. To handle this term we first decompose it into

$$\begin{aligned} &\left\{ \sum_{\substack{(I,J) \in \Omega\mathcal{D}^{\sigma} \times \Omega\mathcal{D}^{\omega}:\ J \subset 3I \\ 2^{-\boldsymbol{\rho}}\ell(I) \leq \ell(J) \leq 2^{\boldsymbol{\rho}}\ell(I)}} + \sum_{\substack{(I,J) \in \Omega\mathcal{D}^{\sigma} \times \Omega\mathcal{D}^{\omega}:\ I \subset 3J \\ 2^{-\boldsymbol{\rho}}\ell(I) \leq \ell(J) \leq 2^{\boldsymbol{\rho}}\ell(I)}} + \sum_{\substack{(I,J) \in \Omega\mathcal{D}^{\sigma} \times \Omega\mathcal{D}^{\omega} \\ 2^{-\boldsymbol{\rho}}\ell(I) \leq \ell(J) \leq 2^{\boldsymbol{\rho}}\ell(I) \\ J \not\subset 3I \text{ and } I \not\subset 3J}} \right\} \left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_I f\right), \triangle^{\omega}_J g\right\rangle_{\omega}\right| \\ &\equiv A_1 + A_2 + A_3. \end{aligned}$$

The proof of the bound for term A_3 is similar to that of the bound for the left side of (6.2), and so we will defer the bound for A_3 until after (6.2) has been proved.

We now consider term A_1 as term A_2 is symmetric. To handle this term we will write the quasiHaar functions h^{σ}_I and h^{ω}_J as linear combinations of the indicators of the children of their supporting quasicubes, denoted I_{θ} and $J_{\theta'}$ respectively. Then we use the quasitesting condition on I_{θ} and $J_{\theta'}$ when they *overlap*, i.e. their interiors intersect; we use the quasiweak boundedness property on I_{θ} and $J_{\theta'}$ when they *touch*, i.e. their interiors are disjoint but their closures intersect (even in just a point); and finally we use the A^{α}_2 condition when I_{θ} and $J_{\theta'}$ are *separated*, i.e. their closures are disjoint. We will suppose initially that the side length of J is at most the side length I, i.e. $\ell(J) \leq \ell(I)$, the proof for $J = \pi I$ being similar but for one

point mentioned below. So suppose that I_θ is a child of I and that $J_{\theta'}$ is a child of J. If $J_{\theta'} \subset I_\theta$ we have from (2.5) that,

$$\begin{aligned}
\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta} \triangle_I^\sigma f\right), \mathbf{1}_{J_{\theta'}} \triangle_J^\omega g\right\rangle_\omega\right| &\lesssim \sup_{a,a'\in\Gamma_n} \frac{\left|\langle f, h_I^{\sigma,a}\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}} \left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right), \mathbf{1}_{J_{\theta'}}\right\rangle_\omega\right| \frac{\left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right|}{\sqrt{|J_{\theta'}|_\omega}} \\
&\lesssim \sup_{a,a'\in\Gamma_n} \frac{\left|\langle f, h_I^{\sigma,a}\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}} \left(\int_{J_{\theta'}} \left|T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right)\right|^2 d\omega\right)^{\frac{1}{2}} \left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right| \\
&\lesssim \sup_{a,a'\in\Gamma_n} \frac{\left|\langle f, h_I^{\sigma,a}\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}} \mathfrak{T}_{T_\alpha} |I_\theta|_\sigma^{\frac{1}{2}} \left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right| \\
&\lesssim \sup_{a,a'\in\Gamma_n} \mathfrak{T}_{T_\alpha} \left|\langle f, h_I^{\sigma,a}\rangle_\sigma\right| \left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right| .
\end{aligned}$$

The point referred to above is that when $J = \pi I$ we write $\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right), \mathbf{1}_{J_{\theta'}}\right\rangle_\omega = \left\langle \mathbf{1}_{I_\theta}, T_\omega^{\alpha,*}\left(\mathbf{1}_{J_{\theta'}}\right)\right\rangle_\sigma$ and get the dual quasitesting constant $\mathfrak{T}_{T_\alpha}^*$. If $J_{\theta'}$ and I_θ touch, then $\ell\left(J_{\theta'}\right) \leq \ell\left(I_\theta\right)$ and we have $J_{\theta'} \subset 3I_\theta \setminus I_\theta$, and so

$$\begin{aligned}
\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta} \triangle_I^\sigma f\right), \mathbf{1}_{J_{\theta'}} \triangle_J^\omega g\right\rangle_\omega\right| &\lesssim \sup_{a,a'\in\Gamma_n} \frac{\left|\langle f, h_I^{\sigma,a}\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}} \left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right), \mathbf{1}_{J_{\theta'}}\right\rangle_\omega\right| \frac{\left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right|}{\sqrt{|J_{\theta'}|_\omega}} \\
&\lesssim \sup_{a,a'\in\Gamma_n} \frac{\left|\langle f, h_I^{\sigma,a}\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}} \mathcal{WBP}_{T^\alpha} \sqrt{|I_\theta|_\sigma |J_{\theta'}|_\omega} \frac{\left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right|}{\sqrt{|J_{\theta'}|_\omega}} \\
&= \sup_{a,a'\in\Gamma_n} \mathcal{WBP}_{T^\alpha} \left|\langle f, h_I^{\sigma,a}\rangle_\sigma\right| \left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right| .
\end{aligned}$$

Finally, if $J_{\theta'}$ and I_θ are separated, and if K is the smallest (not necessarily dyadic) quasicube containing both $J_{\theta'}$ and I_θ, then $\operatorname{dist}\left(I_\theta, J_{\theta'}\right) \approx |K|^{\frac{1}{n}}$ and we have

$$\begin{aligned}
\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta} \triangle_I^\sigma f\right), \mathbf{1}_{J_{\theta'}} \triangle_J^\omega g\right\rangle_\omega\right| &\lesssim \sup_{a,a'\in\Gamma_n} \frac{\left|\langle f, h_I^{\sigma,a}\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}} \left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right), \mathbf{1}_{J_{\theta'}}\right\rangle_\omega\right| \frac{\left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right|}{\sqrt{|J_{\theta'}|_\omega}} \\
&\lesssim \sup_{a,a'\in\Gamma_n} \frac{\left|\langle f, h_I^{\sigma,a}\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}} \frac{1}{\operatorname{dist}\left(I_\theta, J_{\theta'}\right)^{n-\alpha}} |I_\theta|_\sigma |J_{\theta'}|_\omega \frac{\left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right|}{\sqrt{|J_{\theta'}|_\omega}} \\
&= \sup_{a,a'\in\Gamma_n} \frac{\sqrt{|I_\theta|_\sigma |J_{\theta'}|_\omega}}{\operatorname{dist}\left(I_\theta, J_{\theta'}\right)^{n-\alpha}} \left|\langle f, h_I^{\sigma,a}\rangle_\sigma\right| \left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right| \\
&\lesssim \sqrt{A_2^\alpha} \sup_{a,a'\in\Gamma_n} \left|\langle f, h_I^{\sigma,a}\rangle_\sigma\right| \left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right| .
\end{aligned}$$

Now we sum over all the children of J and I satisfying $2^{-\boldsymbol{\rho}}\ell(I) \leq \ell(J) \leq 2^{\boldsymbol{\rho}}\ell(I)$ for which $J \subset 3I$ to obtain that

$$A_1 \lesssim \left(\mathfrak{T}_{T_\alpha} + \mathfrak{T}_{T_\alpha}^* + \mathcal{WBP}_{T^\alpha} + \sqrt{A_2^\alpha}\right) \sup_{a,a'\in\Gamma_n} \sum_{\substack{(I,J)\in\Omega\mathcal{D}^\sigma\times\Omega\mathcal{D}^\omega:\ J\subset 3I \\ 2^{-\boldsymbol{\rho}}\ell(I)\leq\ell(J)\leq 2^{\boldsymbol{\rho}}\ell(I)}} \left|\langle f, h_I^{\sigma,a}\rangle_\sigma\right| \left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right| .$$

Now Cauchy-Schwarz gives the estimate

$$\begin{aligned}
& \sup_{a,a'\in\Gamma_n} \sum_{\substack{(I,J)\in\Omega\mathcal{D}^\sigma\times\Omega\mathcal{D}^\omega:\ J\subset 3I \\ 2^{-\rho}\ell(I)\leq\ell(J)\leq 2^{\rho}\ell(I)}} \left|\langle f,h_I^{\sigma,a}\rangle_\sigma\right| \left|\left\langle g,h_J^{\omega,a'}\right\rangle_\omega\right| \\
\leq & \sup_{a,a'\in\Gamma_n} \left(\sum_{\substack{(I,J)\in\Omega\mathcal{D}^\sigma\times\Omega\mathcal{D}^\omega:\ J\subset 3I \\ 2^{-\rho}\ell(I)\leq\ell(J)\leq 2^{\rho}\ell(I)}} \left|\langle f,h_I^{\sigma}\rangle_\sigma\right|^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{(I,J)\in\Omega\mathcal{D}^\sigma\times\Omega\mathcal{D}^\omega:\ J\subset 3I \\ 2^{-\rho}\ell(I)\leq\ell(J)\leq 2^{\rho}\ell(I)}} \left|\langle g,h_J^{\omega}\rangle_\omega\right|^2 \right)^{\frac{1}{2}} \\
\lesssim & \left\|f\right\|_{L^2(\sigma)} \left\|g\right\|_{L^2(\omega)} ,
\end{aligned}$$

This completes our proof of (6.1) save for the deferral of term A_3, which we bound below.

Now we turn to the sum of separated cubes in (6.2) and (6.4). In each of these inequalities we have either orthogonality or 'quasi' orthogonality, due either to the presence of a quasiHaar projection such as $\triangle_I^\sigma f$, or the presence of an appropriate Carleson indicator such as $\beta_{F,J}\mathbf{1}_F\alpha_{\mathcal{F}}(F)$. We will prove below the estimate for the separated sum corresponding to (6.2). The corresponding estimates for (6.4) are handled in a similar way, the only difference being that the 'quasi' orthogonality of Carleson indicators such as $\beta_{F,J}\mathbf{1}_F\alpha_{\mathcal{F}}(F)$ is used in place of the orthogonality of quasiHaar functions such as $\triangle_I^\sigma f$. The bounded functions $\beta_{F,J}$ are replaced with constants after an application of the energy lemma, and then the arguments proceed as below.

We split the pairs $(I,J)\in\Omega\mathcal{D}^\sigma\times\Omega\mathcal{D}^\omega$ occurring in (6.2) into two groups, those with side length of J smaller than side length of I, and those with side length of I smaller than side length of J, treating only the former case, the latter being symmetric. Thus we prove the following bound:

$$\begin{aligned}
\mathcal{A}(f,g) &\equiv \sum_{\substack{(I,J)\in\Omega\mathcal{D}^\sigma\times\Omega\mathcal{D}^\omega \\ I\cap J=\emptyset \text{ and } \ell(J)\leq 2^{-\rho}\ell(I)}} \left|\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\triangle_J^\omega g\right\rangle_\omega\right| \\
&\lesssim \sqrt{A_2^\alpha}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)} .
\end{aligned}$$

We apply the 'pivotal' bound from the Energy Lemma 8 to estimate the inner product $\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\triangle_J^\omega g\right\rangle_\omega$ and obtain,

$$\left|\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\triangle_J^\omega g\right\rangle_\omega\right| \lesssim \left\|\triangle_J^\omega g\right\|_{L^2(\omega)} \mathrm{P}^\alpha\left(J,\left|\triangle_I^\sigma f\right|\sigma\right)\sqrt{\left|J\right|_\omega} ,$$

Denote by dist the ℓ^∞ distance in $\mathbb{R}^n$: $\operatorname{dist}(x,y)=\max_{1\leq j\leq n}|x_j-y_j|$, and denote the corresponding quasidistance by $\operatorname{qdist}(x,y)=\operatorname{dist}\left(\Omega^{-1}x,\Omega^{-1}y\right)$. We now estimate separately the long-range and mid-range cases where $\operatorname{qdist}(J,I)\geq\ell(I)$ holds or not, and we decompose $\mathcal{A}$ accordingly:

$$\mathcal{A}(f,g)\equiv\mathcal{A}^{\mathrm{long}}(f,g)+\mathcal{A}^{\mathrm{mid}}(f,g) .$$

The long-range case: We begin with the case where $\operatorname{qdist}(J,I)$ is at least $\ell(I)$, i.e. $J\cap 3I=\emptyset$. Since J and I are separated by at least $\max\{\ell(J),\ell(I)\}$, we have the inequality

$$\mathrm{P}^\alpha\left(J,\left|\triangle_I^\sigma f\right|\sigma\right)\approx\int_I\frac{\ell(J)}{\left|y-c_J\right|^{n+1-\alpha}}\left|\triangle_I^\sigma f(y)\right|d\sigma(y)\lesssim\left\|\triangle_I^\sigma f\right\|_{L^2(\sigma)}\frac{\ell(J)\sqrt{\left|I\right|_\sigma}}{\operatorname{qdist}(I,J)^{n+1-\alpha}},$$

since $\int_I |\triangle_I^\sigma f(y)| \, d\sigma(y) \leq \|\triangle_I^\sigma f\|_{L^2(\sigma)} \sqrt{|I|_\sigma}$. Thus with $A(f,g) = \mathcal{A}^{\mathrm{long}}(f,g)$ we have

$$\begin{aligned} A(f,g) &\lesssim \sum_{I\in\Omega\mathcal{D}} \sum_{J\,:\,\ell(J)\leq\ell(I):\ \mathrm{qdist}(I,J)\geq\ell(I)} \|\triangle_I^\sigma f\|_{L^2(\sigma)} \|\triangle_J^\omega g\|_{L^2(\omega)} \\ &\qquad \times \frac{\ell(J)}{\mathrm{qdist}(I,J)^{n+1-\alpha}} \sqrt{|I|_\sigma}\sqrt{|J|_\omega} \\ &\equiv \sum_{(I,J)\in\mathcal{P}} \|\triangle_I^\sigma f\|_{L^2(\sigma)} \|\triangle_J^\omega g\|_{L^2(\omega)} A(I,J); \\ \text{with } A(I,J) &\equiv \frac{\ell(J)}{\mathrm{qdist}(I,J)^{n+1-\alpha}} \sqrt{|I|_\sigma}\sqrt{|J|_\omega}; \\ \text{and } \mathcal{P} &\equiv \{(I,J)\in\Omega\mathcal{D}\times\Omega\mathcal{D} : \ell(J)\leq\ell(I) \text{ and } \mathrm{qdist}(I,J)\geq\ell(I)\}. \end{aligned}$$

Now let $\Omega\mathcal{D}_N \equiv \left\{K\in\Omega\mathcal{D} : \ell(K) = 2^N\right\}$ for each $N\in\mathbb{Z}$. For $N\in\mathbb{Z}$ and $s\in\mathbb{Z}_+$, we further decompose $A(f,g)$ by pigeonholing the sidelengths of I and J by 2^N and 2^{N-s} respectively:

$$\begin{aligned} A(f,g) &= \sum_{s=0}^{\infty}\sum_{N\in\mathbb{Z}} A_N^s(f,g); \\ A_N^s(f,g) &\equiv \sum_{(I,J)\in\mathcal{P}_N^s} \|\triangle_I^\sigma f\|_{L^2(\sigma)} \|\triangle_J^\omega g\|_{L^2(\omega)} A(I,J) \\ \text{where } \mathcal{P}_N^s &\equiv \{(I,J)\in\Omega\mathcal{D}_N\times\Omega\mathcal{D}_{N-s} : \mathrm{qdist}(I,J)\geq\ell(I)\}. \end{aligned}$$

Now $A_N^s(f,g) = A_N^s\left(\mathsf{P}_N^\sigma f, \mathsf{P}_{N-s}^\omega g\right)$ where $\mathsf{P}_M^\mu = \sum_{K\in\Omega\mathcal{D}_M} \triangle_K^\mu$ denotes quasiHaar projection onto Span $\{h_K^{\mu,a}\}_{K\in\Omega\mathcal{D}_M, a\in\Gamma_n}$, and so by orthogonality of the projections $\{\mathsf{P}_M^\mu\}_{M\in\mathbb{Z}}$ we have

$$\begin{aligned} \left|\sum_{N\in\mathbb{Z}} A_N^s(f,g)\right| &= \sum_{N\in\mathbb{Z}} \left|A_N^s\left(\mathsf{P}_N^\sigma f, \mathsf{P}_{N-s}^\omega g\right)\right| \leq \sum_{N\in\mathbb{Z}} \|A_N^s\| \, \|\mathsf{P}_N^\sigma f\|_{L^2(\sigma)} \left\|\mathsf{P}_{N-s}^\omega g\right\|_{L^2(\omega)} \\ &\leq \left\{\sup_{N\in\mathbb{Z}} \|A_N^s\|\right\} \left(\sum_{N\in\mathbb{Z}} \|\mathsf{P}_N^\sigma f\|_{L^2(\sigma)}^2\right)^{\frac{1}{2}} \left(\sum_{N\in\mathbb{Z}} \left\|\mathsf{P}_{N-s}^\omega g\right\|_{L^2(\omega)}^2\right)^{\frac{1}{2}} \\ &\leq \left\{\sup_{N\in\mathbb{Z}} \|A_N^s\|\right\} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}. \end{aligned}$$

Thus it suffices to show an estimate uniform in N with geometric decay in s, and we will show

$$|A_N^s(f,g)| \leq C2^{-s}\sqrt{A_2^\alpha}\, \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}, \qquad \text{for } s\geq 0 \text{ and } N\in\mathbb{Z}. \tag{6.5}$$

We now pigeonhole the distance between I and J:

$$\begin{aligned} A_N^s(f,g) &= \sum_{\ell=0}^{\infty} A_{N,\ell}^s(f,g); \\ A_{N,\ell}^s(f,g) &\equiv \sum_{(I,J)\in\mathcal{P}_{N,\ell}^s} \|\triangle_I^\sigma f\|_{L^2(\sigma)} \|\triangle_J^\omega g\|_{L^2(\omega)} A(I,J) \\ \text{where } \mathcal{P}_{N,\ell}^s &\equiv \left\{(I,J)\in\Omega\mathcal{D}_N\times\Omega\mathcal{D}_{N-s} : \mathrm{qdist}(I,J)\approx 2^{N+\ell}\right\}. \end{aligned}$$

If we define $\mathcal{H}\left(A_{N,\ell}^s\right)$ to be the bilinear form on $\ell^2\times\ell^2$ with matrix $[A(I,J)]_{(I,J)\in\mathcal{P}_{N,\ell}^s}$, then it remains to show that the norm $\left\|\mathcal{H}\left(A_{N,\ell}^s\right)\right\|_{\ell^2\to\ell^2}$ of $\mathcal{H}\left(A_{N,\ell}^s\right)$ on the sequence space ℓ^2 is bounded by $C2^{-s-\ell}\sqrt{A_2^\alpha}$. In turn, this is equivalent to showing that the norm $\left\|\mathcal{H}\left(B_{N,\ell}^s\right)\right\|_{\ell^2\to\ell^2}$ of the bilinear form $\mathcal{H}\left(B_{N,\ell}^s\right) \equiv \mathcal{H}\left(A_{N,\ell}^s\right)^{\mathrm{tr}}\mathcal{H}\left(A_{N,\ell}^s\right)$ on the sequence space ℓ^2 is bounded by $C^2 2^{-2s-2\ell}A_2^\alpha$. Here $\mathcal{H}\left(B_{N,\ell}^s\right)$ is the quadratic

form with matrix kernel $\left[B^s_{N,\ell}(J,J')\right]_{J,J'\in\Omega\mathcal{D}_{N-s}}$ having entries:

$$B^s_{N,\ell}(J,J') \equiv \sum_{I\in\Omega\mathcal{D}_N:\ \mathrm{qdist}(I,J)\approx\mathrm{qdist}(I,J')\approx 2^{N+\ell}} A(I,J)\,A(I,J'), \qquad \text{for } J,J'\in\Omega\mathcal{D}_{N-s}.$$

We are reduced to showing,

$$\left\|\mathcal{H}\left(B^s_{N,\ell}\right)\right\|_{\ell^2\to\ell^2} \leq C2^{-2s-2\ell}A_2^\alpha \quad \text{for } s\geq 0,\ \ell\geq 0 \text{ and } N\in\mathbb{Z}.$$

We begin by computing $B^s_{N,\ell}(J,J')$:

$$\begin{aligned} B^s_{N,\ell}(J,J') &= \sum_{\substack{I\in\Omega\mathcal{D}_N\\ \mathrm{qdist}(I,J)\approx\mathrm{qdist}\left(I,J'\right)\approx 2^{N+\ell}}} \frac{\ell(J)}{\mathrm{qdist}(I,J)^{n+1-\alpha}}\sqrt{|I|_\sigma}\sqrt{|J|_\omega} \\ &\qquad\times\frac{\ell(J')}{\mathrm{qdist}(I,J')^{n+1-\alpha}}\sqrt{|I|_\sigma}\sqrt{|J'|_\omega} \\ &= \left\{\sum_{\substack{I\in\Omega\mathcal{D}_N\\ \mathrm{qdist}(I,J)\approx\mathrm{qdist}\left(I,J'\right)\approx 2^{N+\ell}}} |I|_\sigma\,\frac{1}{\mathrm{qdist}(I,J)^{n+1-\alpha}\,\mathrm{qdist}(I,J')^{n+1-\alpha}}\right\} \\ &\qquad\times\ell(J)\,\ell(J')\sqrt{|J|_\omega}\sqrt{|J'|_\omega}. \end{aligned}$$

Now we show that

$$\left\|B^s_{N,\ell}\right\|_{\ell^2\to\ell^2} \lesssim 2^{-2s-2\ell}A_2^\alpha\ , \tag{6.6}$$

by applying the proof of Schur's lemma. Fix $\ell\geq 0$ and $s\geq 0$. Choose the Schur function $\beta(K)=\frac{1}{\sqrt{|K|_\omega}}$. Fix $J\in\Omega\mathcal{D}_{N-s}$. We now group those $I\in\Omega\mathcal{D}_N$ with $\mathrm{qdist}(I,J)\approx 2^{N+\ell}$ into finitely many groups $G_1,...G_{C_n}$ for which the union of the I in each group is contained in quasicube of side length roughly $\frac{1}{100}2^{N+\ell}$, and we set $I_k^*\equiv\bigcup_{I\in G_k} I$ for $1\leq k\leq C_n$. We then have

$$\begin{aligned} &\sum_{J'\in\Omega\mathcal{D}_{N-s}} \frac{\beta(J)}{\beta(J')}B^s_{N,\ell}(J,J') \\ &= \sum_{\substack{J'\in\Omega\mathcal{D}_{N-s}\\ \mathrm{qdist}\left(J',J\right)\leq\frac{1}{100}2^{N+\ell+2}}} \frac{\beta(J)}{\beta(J')}B^s_{N,\ell}(J,J') + \sum_{\substack{J'\in\Omega\mathcal{D}_{N-s}\\ \mathrm{qdist}\left(J',J\right)>\frac{1}{100}2^{N+\ell+2}}} \frac{\beta(J)}{\beta(J')}B^s_{N,\ell}(J,J') \\ &= A+B, \end{aligned}$$

where

$$\begin{aligned}
A &\lesssim \sum_{\substack{J'\in\Omega\mathcal{D}_{N-s}\\ \operatorname{qdist}(J,J')\leq\frac{1}{100}2^{N+\ell+2}}} \left\{\sum_{\substack{I\in\Omega\mathcal{D}_N\\ \operatorname{qdist}(I,J)\approx 2^{N+\ell}}} |I|_\sigma\right\} \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}} |J'|_\omega \\
&= \sum_{\substack{J'\in\Omega\mathcal{D}_{N-s}\\ \operatorname{qdist}(J,J')\leq\frac{1}{100}2^{N+\ell+2}}} \left\{\sum_{k=1}^{C_n} |I_k^*|_\sigma\right\} \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}} |J'|_\omega \\
&= \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}} \sum_{k=1}^{C_n} \sum_{\substack{J'\in\Omega\mathcal{D}_{N-s}\\ \operatorname{qdist}(J,J')\leq\frac{1}{100}2^{N+\ell+2}}} |I_k^*|_\sigma \ |J'|_\omega \\
&\lesssim 2^{-2s-2\ell} \sum_{k=1}^{C_n} \frac{|I_k^*|_\sigma}{2^{(\ell+N)(n-\alpha)}} \frac{\left|\frac{1}{100}2^{N+\ell+2}J\right|_\omega}{2^{(\ell+N)(n-\alpha)}} \lesssim 2^{-2s-2\ell} A_2^\alpha,
\end{aligned}$$

since the quasicubes I_k^* and $\frac{1}{100}2^{N+\ell+2}J$ are well separated. If we let Q_k be the smallest quasicube containing the set

$$E_k \equiv \bigcup_{\substack{J'\in\Omega\mathcal{D}_{N-s}:\ \operatorname{qdist}(I_k^*,J')\approx 2^{N+\ell}\\ \operatorname{qdist}(J,J')>\frac{1}{100}2^{N+\ell+2}}} J' \ ,$$

we also have

$$\begin{aligned}
B &\lesssim \sum_{\substack{J'\in\Omega\mathcal{D}_{N-s}\\ \operatorname{qdist}(J,J')>\frac{1}{100}2^{N+\ell+2}}} \left\{\sum_{\substack{I\in\Omega\mathcal{D}_N\\ \operatorname{qdist}(I,J')\approx\operatorname{qdist}(I,J)\approx 2^{N+\ell}}} |I|_\sigma\right\} \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}} |J'|_\omega \\
&\lesssim \sum_{\substack{J'\in\Omega\mathcal{D}_{N-s}\\ \operatorname{qdist}(J,J')>\frac{1}{100}2^{N+\ell+2}}} \left\{\sum_{k:\ \operatorname{qdist}(I_k^*,J')\approx 2^{N+\ell}} |I_k^*|_\sigma\right\} \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}} |J'|_\omega \\
&\lesssim \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}} \sum_{k=1}^{C_n} |I_k^*|_\sigma \, |E_k|_\omega \\
&\lesssim 2^{-2s-2\ell} \sum_{k=1}^{C_n} \frac{|I_k^*|_\sigma}{2^{(\ell+N)(n-\alpha)}} \frac{|Q_k|_\omega}{2^{(\ell+N)(n-\alpha)}} \lesssim 2^{-2s-2\ell} A_2^\alpha,
\end{aligned}$$

since the quasicube I_k^* is well separated from the quasicube Q_k.

Thus we can now apply Schur's argument with $\sum_J (a_J)^2 = \sum_{J'} (b_{J'})^2 = 1$ to obtain

$$\begin{aligned}
&\sum_{J,J' \in \Omega\mathcal{D}_{N-s}} a_J b_{J'} B^s_{N,\ell}(J,J') \\
=& \sum_{J,J' \in \Omega\mathcal{D}_{N-s}} a_J \beta(J) b_{J'} \beta(J') \frac{B^s_{N,\ell}(J,J')}{\beta(J)\beta(J')} \\
\leq& \sum_J (a_J \beta(J))^2 \sum_{J'} \frac{B^s_{N,\ell}(J,J')}{\beta(J)\beta(J')} + \sum_{J'} (b_{J'}\beta(J'))^2 \sum_J \frac{B^s_{N,\ell}(J,J')}{\beta(J)\beta(J')} \\
=& \sum_J (a_J)^2 \left\{ \sum_{J'} \frac{\beta(J)}{\beta(J')} B^s_{N,\ell}(J,J') \right\} + \sum_{J'} (b_{J'})^2 \left\{ \sum_J \frac{\beta(J')}{\beta(J)} B^s_{N,\ell}(J,J') \right\} \\
\lesssim& 2^{-2s-2\ell} A^\alpha_2 \left(\sum_J (a_J)^2 + \sum_{J'} (b_{J'})^2 \right) = 2^{1-2s-2\ell} A^\alpha_2 .
\end{aligned}$$

This completes the proof of (6.6). We can now sum in ℓ to get (6.5) and we are done. This completes our proof of the long-range estimate

$$\mathcal{A}^{\mathrm{long}}(f,g) \lesssim \sqrt{A^\alpha_2} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} \ .$$

At this point we pause to complete the proof of (6.1). Indeed, the deferred term A_3 can be handled using the above argument since $3J \cap I = \emptyset = J \cap 3I$ implies that we can use the Energy Lemma 8 as we did above.

The mid range case: Let

$$\mathcal{P} \equiv \left\{ (I,J) \in \Omega\mathcal{D} \times \Omega\mathcal{D} : J \text{ is good}, \ \ell(J) \leq 2^{-\boldsymbol{\rho}} \ell(I), \ J \subset 3I \setminus I \right\}.$$

For $(I,J) \in \mathcal{P}$, the 'pivotal' estimate from the Energy Lemma 8 gives

$$\left| \left\langle T^\alpha_\sigma (\triangle^\sigma_I f), \triangle^\omega_J g \right\rangle_\omega \right| \lesssim \|\triangle^\omega_J g\|_{L^2(\omega)} \mathrm{P}^\alpha (J, |\triangle^\sigma_I f| \sigma) \sqrt{|J|_\omega} .$$

Now we pigeonhole the lengths of I and J and the distance between them by defining

$$\mathcal{P}^s_{N,d} \equiv \left\{ (I,J) \in \Omega\mathcal{D} \times \Omega\mathcal{D} : J \text{ is good}, \ \ell(I) = 2^N, \ \ell(J) = 2^{N-s}, \ J \subset 3I \setminus I, \ 2^{d-1} \leq \mathrm{qdist}(I,J) \leq 2^d \right\}.$$

Note that the closest a good quasicube J can come to I is determined by the goodness inequality, which gives this bound for $2^d \geq \mathrm{qdist}(I,J)$:

$$\begin{aligned}
& 2^d \geq \frac{1}{2} \ell(I)^{1-\varepsilon} \ell(J)^\varepsilon = \frac{1}{2} 2^{N(1-\varepsilon)} 2^{(N-s)\varepsilon} = \frac{1}{2} 2^{N-\varepsilon s}; \\
& \text{which implies } N - \varepsilon s - 1 \leq d \leq N,
\end{aligned}$$

where the last inequality holds because we are in the case of the mid-range term. Thus we have

$$\begin{aligned}
\sum_{(I,J)\in\mathcal{P}} \left| \left\langle T^\alpha_\sigma (\triangle^\sigma_I f), \triangle^\omega_J g \right\rangle_\omega \right| &\lesssim \sum_{(I,J)\in\mathcal{P}} \|\triangle^\omega_J g\|_{L^2(\omega)} \mathrm{P}^\alpha (J, |\triangle^\sigma_I f| \sigma) \sqrt{|J|_\omega} \\
&= \sum_{s=\boldsymbol{\rho}}^\infty \sum_{N\in\mathbb{Z}} \sum_{d=N-\varepsilon s-1}^{N} \sum_{(I,J)\in\mathcal{P}^s_{N,d}} \|\triangle^\omega_J g\|_{L^2(\omega)} \mathrm{P}^\alpha (J, |\triangle^\sigma_I f| \sigma) \sqrt{|J|_\omega}.
\end{aligned}$$

Now we use

$$\begin{aligned}
\mathrm{P}^\alpha (J, |\triangle^\sigma_I f| \sigma) &= \int_I \frac{\ell(J)}{(\ell(J) + |y - c_J|)^{n+1-\alpha}} |\triangle^\sigma_I f(y)| \, d\sigma(y) \\
&\lesssim \frac{2^{N-s}}{2^{d(n+1-\alpha)}} \|\triangle^\sigma_I f\|_{L^2(\sigma)} \sqrt{|I|_\sigma}
\end{aligned}$$

and apply Cauchy-Schwarz in J and use $J \subset 3I \setminus I$ to get

$$\begin{aligned}
&\sum_{(I,J)\in\mathcal{P}} \left|\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\triangle_J^\omega g\right\rangle_\omega\right| \\
\lesssim\ & \sum_{s=\boldsymbol{\rho}}^{\infty}\sum_{N\in\mathbb{Z}}\sum_{d=N-\varepsilon s-1}^{N}\sum_{I\in\Omega\mathcal{D}_N}\frac{2^{N-s}2^{N(n-\alpha)}}{2^{d(n+1-\alpha)}}\left\|\triangle_I^\sigma f\right\|_{L^2(\sigma)}\frac{\sqrt{|I|_\sigma}\sqrt{|3I\setminus I|_\omega}}{2^{N(n-\alpha)}} \\
&\qquad\qquad\times\sqrt{\sum_{\substack{J\in\Omega\mathcal{D}_{N-s}\\ J\subset 3I\setminus I\text{ and }\operatorname{qdist}(I,J)\approx 2^d}}\left\|\triangle_J^\omega g\right\|_{L^2(\omega)}^2} \\
\lesssim\ & (1+\varepsilon s)\sum_{s=\boldsymbol{\rho}}^{\infty}\sum_{N\in\mathbb{Z}}\frac{2^{N-s}2^{N(n-\alpha)}}{2^{(N-\varepsilon s)(n+1-\alpha)}}\sqrt{A_2^\alpha}\sum_{I\in\Omega\mathcal{D}_N}\left\|\triangle_I^\sigma f\right\|_{L^2(\sigma)}\sqrt{\sum_{\substack{J\in\Omega\mathcal{D}_{N-s}\\ J\subset 3I\setminus I}}\left\|\triangle_J^\omega g\right\|_{L^2(\omega)}^2} \\
\lesssim\ & (1+\varepsilon s)\sum_{s=\boldsymbol{\rho}}^{\infty}2^{-s[1-\varepsilon(n+1-\alpha)]}\sqrt{A_2^\alpha}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}\lesssim\sqrt{A_2^\alpha}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)},
\end{aligned}$$

where in the third line above we have used $\sum_{d=N-\varepsilon s-1}^{N}1\lesssim 1+\varepsilon s$, and in the last line $\frac{2^{N-s}2^{N(n-\alpha)}}{2^{(N-\varepsilon s)(n+1-\alpha)}}=2^{-s[1-\varepsilon(n+1-\alpha)]}$ followed by Cauchy-Schwarz in I and N, using that we have bounded overlap in the triples of I for $I\in\Omega\mathcal{D}_N$. More precisely, if we define $f_k\equiv\sum_{I\in\Omega\mathcal{D}_k}\triangle_I^\sigma f h_I^\sigma$ and $g_k\equiv\sum_{I\in\Omega\mathcal{D}_k}\triangle_J^\omega g h_J^\omega$, then we have the orthogonality inequality

$$\begin{aligned}
\sum_{N\in\mathbb{Z}}\left\|f_N\right\|_{L^2(\sigma)}\left\|g_{N-s}\right\|_{L^2(\omega)} &\leq \left(\sum_{N\in\mathbb{Z}}\left\|f_N\right\|_{L^2(\sigma)}^2\right)^{\frac{1}{2}}\left(\sum_{N\in\mathbb{Z}}\left\|g_{N-s}\right\|_{L^2(\omega)}^2\right)^{\frac{1}{2}} \\
&= \left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}.
\end{aligned}$$

We have assumed that $0<\varepsilon<\frac{1}{n+1-\alpha}$ in the calculations above, and this completes the proof of Lemma 9. $\square$

7. Corona Decompositions and splittings

We will use two different corona constructions, namely a Calderón-Zygmund decomposition and an energy decomposition of NTV type, to reduce matters to the stopping form, the main part of which is handled by Lacey's recursion argument. We will then iterate these coronas into a double corona. We first recall our basic setup. For convenience in notation we will sometimes suppress the dependence on α in our nonlinear forms, but will retain it in the operators, Poisson integrals and constants. We will assume that the good/bad quasicube machinery of Nazarov, Treil and Volberg [Vol] is in force here. Let $\Omega\mathcal{D}^\sigma=\Omega\mathcal{D}^\omega$ be an $(\mathbf{r},\varepsilon)$-good quasigrid on $\mathbb{R}^n$, and let $\{h_I^{\sigma,a}\}_{I\in\Omega\mathcal{D}^\sigma,\ a\in\Gamma_n}$ and $\left\{h_J^{\omega,b}\right\}_{J\in\Omega\mathcal{D}^\omega,\ b\in\Gamma_n}$ be corresponding quasiHaar bases as described above, so that

$$f=\sum_{I\in\Omega\mathcal{D}^\sigma}\triangle_I^\sigma f\text{ and }g=\sum_{J\in\Omega\mathcal{D}^\omega}\triangle_J^\omega g\ ,$$

where the quasiHaar projections $\triangle_I^\sigma f$ and $\triangle_J^\omega g$ vanish if the quasicubes I and J are not good. Inequality (2.8) is equivalent to boundedness of the bilinear form

$$\mathcal{T}^\alpha(f,g)\equiv\left\langle T_\sigma^\alpha(f),g\right\rangle_\omega=\sum_{I\in\Omega\mathcal{D}^\sigma\text{ and }J\in\Omega\mathcal{D}^\omega}\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\triangle_J^\omega g\right\rangle_\omega$$

on $L^2(\sigma)\times L^2(\omega)$, i.e.

$$\left|\mathcal{T}^\alpha(f,g)\right|\leq\mathfrak{N}_{T^\alpha}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}.$$

7.1. **The Calderón-Zygmund corona.** We now introduce a stopping tree $\mathcal{F}$ for the function $f \in L^2(\sigma)$. Let $\mathcal{F}$ be a collection of Calderón-Zygmund stopping quasicubes for f, and let $\Omega\mathcal{D}^\sigma = \bigcup_{F\in\mathcal{F}} \mathcal{C}_F$ be the associated corona decomposition of the dyadic quasigrid $\Omega\mathcal{D}^\sigma$.

For a quasicube $I \in \Omega\mathcal{D}^\sigma$ let $\pi_{\Omega\mathcal{D}^\sigma} I$ be the $\Omega\mathcal{D}^\sigma$-parent of I in the quasigrid $\Omega\mathcal{D}^\sigma$, and let $\pi_{\mathcal{F}} I$ be the smallest member of $\mathcal{F}$ that contains I. For $F, F' \in \mathcal{F}$, we say that F' is an $\mathcal{F}$-child of F if $\pi_{\mathcal{F}}(\pi_{\Omega\mathcal{D}^\sigma} F') = F$ (it could be that $F = \pi_{\Omega\mathcal{D}^\sigma} F'$), and we denote by $\mathfrak{C}_{\mathcal{F}}(F)$ the set of $\mathcal{F}$-children of F. For $F \in \mathcal{F}$, define the projection $\mathsf{P}^\sigma_{\mathcal{C}_F}$ onto the linear span of the quasiHaar functions $\{h_I^{\sigma,a}\}_{I\in\mathcal{C}_F,\ a\in\Gamma_n}$ by

$$\mathsf{P}^\sigma_{\mathcal{C}_F} f = \sum_{I\in\mathcal{C}_F} \triangle^\sigma_I f = \sum_{I\in\mathcal{C}_F,\ a\in\Gamma_n} \left\langle f, h_I^{\sigma,a}\right\rangle_\sigma h_I^{\sigma,a}.$$

The standard properties of these projections are

$$f = \sum_{F\in\mathcal{F}} \mathsf{P}^\sigma_{\mathcal{C}_F} f, \qquad \int \left(\mathsf{P}^\sigma_{\mathcal{C}_F} f\right)\sigma = 0, \qquad \|f\|^2_{L^2(\sigma)} = \sum_{F\in\mathcal{F}} \left\|\mathsf{P}^\sigma_{\mathcal{C}_F} f\right\|^2_{L^2(\sigma)}.$$

7.2. **The energy corona.** We must also impose a quasienergy corona decomposition as in [NTV4] and [LaSaUr2].

Definition 9. *Given a quasicube S_0, define $\mathcal{S}(S_0)$ to be the maximal subquasicubes $I \subset S_0$ such that*

$$\sum_{J\in\mathcal{M}_{\boldsymbol{\tau}-\mathrm{deep}}(I)} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{S_0\setminus\gamma J}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\mathrm{subgood},\omega}_J \mathbf{x}\right\|^2_{L^2(\omega)} \geq C_{\mathrm{energy}}\left[\left(\mathcal{E}^{\mathrm{deep}}_\alpha\right)^2 + A^\alpha_2 + A^{\alpha,\mathrm{punct}}_2\right] |I|_\sigma, \tag{7.1}$$

where $\mathcal{E}^{\mathrm{deep}}_\alpha$ is the constant in the deep quasienergy condition defined in Definition 8, and C_{energy} is a sufficiently large positive constant depending only on $\boldsymbol{\tau} \geq \mathbf{r}, n$ and α. Then define the σ-energy stopping quasicubes of S_0 to be the collection

$$\mathcal{S} = \{S_0\} \cup \bigcup_{n=0}^\infty \mathcal{S}_n$$

where $\mathcal{S}_0 = \mathcal{S}(S_0)$ and $\mathcal{S}_{n+1} = \bigcup_{S\in\mathcal{S}_n} \mathcal{S}(S)$ for $n \geq 0$.

From the quasienergy condition in Definition 8 we obtain the σ-Carleson estimate

$$\sum_{S\in\mathcal{S}:\ S\subset I} |S|_\sigma \leq 2\,|I|_\sigma, \qquad I \in \Omega\mathcal{D}^\sigma. \tag{7.2}$$

Indeed, using the deep quasienergy condition, the first generation satisfies

$$\begin{aligned}
&\sum_{S\in\mathcal{S}_1} |S|_\sigma \\
\leq\;& \frac{1}{C_{\mathrm{energy}}\left[\left(\mathcal{E}^{\mathrm{deep}}_\alpha\right)^2 + A^\alpha_2\right]} \sum_{S\in\mathcal{S}_1}\sum_{J\in\mathcal{M}_{\boldsymbol{\tau}-\mathrm{deep}}(S)} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{S_0\setminus\gamma J}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\mathrm{subgood},\omega}_J \mathbf{x}\right\|^2_{L^2(\omega)} \\
\leq\;& \frac{1}{C_{\mathrm{energy}}\left[\left(\mathcal{E}^{\mathrm{deep}}_\alpha\right)^2 + A^\alpha_2\right]} \sum_{S\in\mathcal{S}_1}\sum_{J\in\mathcal{M}_{\boldsymbol{\tau}-\mathrm{deep}}(S)} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{S_0}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\mathrm{subgood},\omega}_J \mathbf{x}\right\|^2_{L^2(\omega)} \\
\leq\;& \frac{C_{\boldsymbol{\tau},\mathbf{r},n,\alpha}}{C_{\mathrm{energy}}\left[\left(\mathcal{E}^{\mathrm{deep}}_\alpha\right)^2 + A^\alpha_2\right]} \sum_{S\in\mathcal{S}_1}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(S)} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{S_0}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\mathrm{subgood},\omega}_J \mathbf{x}\right\|^2_{L^2(\omega)} \\
\leq\;& \frac{C_{\boldsymbol{\tau},\mathbf{r},n,\alpha}}{C_{\mathrm{energy}}\left[\left(\mathcal{E}^{\mathrm{deep}}_\alpha\right)^2 + A^\alpha_2 + A^{\alpha,\mathrm{punct}}_2\right]} \left(\mathcal{E}^{\mathrm{deep\,plug}}_\alpha\right)^2 |S_0|_\sigma = \frac{1}{2}|S_0|_\sigma\ ,
\end{aligned} \tag{7.3}$$

provided we take $C_{\text{energy}} = 2C_{\boldsymbol{\tau},\mathbf{r},n,\alpha}\frac{\left(\mathcal{E}_{\alpha}^{\text{deep plug}}\right)^2}{\left(\mathcal{E}_{\alpha}^{\text{deep}}\right)^2+A_2^{\alpha}}$ and where from Corollary 1 we have $\mathcal{E}_{\alpha}^{\text{deep plug}} \lesssim \mathcal{E}_{\alpha}^{\text{deep}} + \sqrt{A_2^{\alpha}}+\sqrt{A_2^{\alpha,\text{punct}}}$. The third inequality above, in which $\boldsymbol{\tau}$ is replaced by $\mathbf{r}$ (but the goodness parameter $\varepsilon > 0$ is unchanged), follows because if $J_1 \in \mathcal{M}_{\boldsymbol{\tau}-\text{deep}}(S)$, then $J_1 \subset J_2$ for a unique $J_2 \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(S)$ and we have $\ell(J_2) \leq 2^{\boldsymbol{\tau}-\mathbf{r}}\ell(J_1)$ from the definitions of $\mathcal{M}_{\boldsymbol{\tau}-\text{deep}}(S)$ and $\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(S)$, hence $\frac{\mathrm{P}^{\alpha}\left(J_1,\mathbf{1}_{S_0}\sigma\right)}{|J_1|^{\frac{1}{n}}} \leq C_{\boldsymbol{\tau},\mathbf{r},n,\alpha}\frac{\mathrm{P}^{\alpha}\left(J_2,\mathbf{1}_{S_0}\sigma\right)}{|J_2|^{\frac{1}{n}}}$. Subsequent generations satisfy a similar estimate, which then easily gives (7.2). We emphasize that this collection of stopping times depends only on S_0 and the weight pair (σ,ω), and not on any functions at hand.

Finally, we record the reason for introducing quasienergy stopping times. If

$$X_{\alpha}(\mathcal{C}_S)^2 \equiv \sup_{I\in\mathcal{C}_S}\frac{1}{|I|_{\sigma}}\sum_{J\in\mathcal{M}_{\boldsymbol{\tau}-\text{deep}}(I)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{S\setminus\gamma J}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_J^{\text{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \tag{7.4}$$

is (the square of) the α-*stopping quasienergy* of the weight pair (σ,ω) with respect to the corona $\mathcal{C}_S$, then we have the *stopping quasienergy bounds*

$$X_{\alpha}(\mathcal{C}_S) \leq \sqrt{C_{\text{energy}}}\sqrt{\left(\mathcal{E}_{\alpha}^{\text{deep}}\right)^2 + A_2^{\alpha} + A_2^{\alpha,\text{punct}}}, \qquad S\in\mathcal{S}, \tag{7.5}$$

where A_2^{α}, $A_2^{\alpha,\text{punct}}$ and the the deep quasienergy constant $\mathcal{E}_{\alpha}^{\text{deep}}$ are controlled by assumption.

7.3. **General stopping data.** It is useful to extend our notion of corona decomposition to more general stopping data. Our general definition of stopping data will use a positive constant $C_0 \geq 4$.

Definition 10. *Suppose we are given a positive constant $C_0 \geq 4$, a subset $\mathcal{F}$ of the dyadic quasigrid $\Omega\mathcal{D}^{\sigma}$ (called the stopping times), and a corresponding sequence $\alpha_{\mathcal{F}} \equiv \{\alpha_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$ of nonnegative numbers $\alpha_{\mathcal{F}}(F) \geq 0$ (called the stopping data). Let $(\mathcal{F},\prec,\pi_{\mathcal{F}})$ be the tree structure on $\mathcal{F}$ inherited from $\Omega\mathcal{D}^{\sigma}$, and for each $F\in\mathcal{F}$ denote by $\mathcal{C}_F = \{I\in\Omega\mathcal{D}^{\sigma} : \pi_{\mathcal{F}}I = F\}$ the corona associated with F:*

$$\mathcal{C}_F = \left\{I\in\Omega\mathcal{D}^{\sigma} : I\subset F \text{ and } I\not\subset F' \text{ for any } F'\prec F\right\}.$$

We say the triple $(C_0,\mathcal{F},\alpha_{\mathcal{F}})$ constitutes stopping data *for a function $f\in L^1_{loc}(\sigma)$ if*

1. $\mathbb{E}_I^{\sigma}|f| \leq \alpha_{\mathcal{F}}(F)$ *for all* $I\in\mathcal{C}_F$ *and* $F\in\mathcal{F}$,
2. $\sum_{F'\preceq F}|F'|_{\sigma} \leq C_0|F|_{\sigma}$ *for all* $F\in\mathcal{F}$,
3. $\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)^2|F|_{\sigma} \leq C_0^2\|f\|_{L^2(\sigma)}^2$,
4. $\alpha_{\mathcal{F}}(F) \leq \alpha_{\mathcal{F}}(F')$ *whenever* $F',F\in\mathcal{F}$ *with* $F'\subset F$.

Definition 11. *If $(C_0,\mathcal{F},\alpha_{\mathcal{F}})$ constitutes (general)* stopping data *for a function $f\in L^1_{loc}(\sigma)$, we refer to the othogonal decomposition*

$$f = \sum_{F\in\mathcal{F}}\mathsf{P}_{\mathcal{C}_F}^{\sigma}f; \qquad \mathsf{P}_{\mathcal{C}_F}^{\sigma}f \equiv \sum_{I\in\mathcal{C}_F}\triangle_I^{\sigma}f,$$

as the (general) corona decomposition *of f associated with the stopping times $\mathcal{F}$.*

Property (1) says that $\alpha_{\mathcal{F}}(F)$ bounds the quasiaverages of f in the corona $\mathcal{C}_F$, and property (2) says that the quasicubes at the tops of the coronas satisfy a Carleson condition relative to the weight σ. Note that a standard 'maximal quasicube' argument extends the Carleson condition in property (2) to the inequality

$$\sum_{F'\in\mathcal{F}:\ F'\subset A}|F'|_{\sigma} \leq C_0|A|_{\sigma} \text{ for all open sets } A\subset\mathbb{R}^n.$$

Property (3) is the 'quasi' orthogonality condition that says the sequence of functions $\{\alpha_{\mathcal{F}}(F)\mathbf{1}_F\}_{F\in\mathcal{F}}$ is in the vector-valued space $L^2\left(\ell^2;\sigma\right)$, and property (4) says that the control on stopping data is nondecreasing on the stopping tree $\mathcal{F}$. We emphasize that we are *not* assuming in this definition the stronger property that there is $C>1$ such that $\alpha_{\mathcal{F}}(F') > C\alpha_{\mathcal{F}}(F)$ whenever $F',F\in\mathcal{F}$ with $F'\subsetneqq F$. Instead, the properties (2)

and (3) substitute for this lack. Of course the stronger property *does* hold for the familiar *Calderón-Zygmund* stopping data determined by the following requirements for $C>1$,

$$\begin{aligned}\mathbb{E}_{F'}^{\sigma}|f| &> C\mathbb{E}_{F}^{\sigma}|f| \text{ whenever } F',F\in\mathcal{F} \text{ with } F'\subsetneqq F,\\ \mathbb{E}_{I}^{\sigma}|f| &\leq C\mathbb{E}_{F}^{\sigma}|f| \text{ for } I\in\mathcal{C}_F,\end{aligned}$$

which are themselves sufficiently strong to automatically force properties (2) and (3) with $\alpha_{\mathcal{F}}(F)=\mathbb{E}_{F}^{\sigma}|f|$.

We have the following useful consequence of (2) and (3) that says the sequence $\{\alpha_{\mathcal{F}}(F)\mathbf{1}_F\}_{F\in\mathcal{F}}$ has a *'quasi' orthogonal* property relative to f with a constant C_0' depending only on C_0:

$$\left\|\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)\mathbf{1}_F\right\|_{L^2(\sigma)}^2\leq C_0'\|f\|_{L^2(\sigma)}^2. \tag{7.6}$$

Indeed, the Carleson condition (2) implies a geometric decay in levels of the tree $\mathcal{F}$, namely that there are positive constants C_1 and ε, depending on C_0, such that if $\mathfrak{C}_{\mathcal{F}}^{(n)}(F)$ denotes the set of n^{th} generation children of F in $\mathcal{F}$,

$$\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F):}|F'|_\sigma\leq\left(C_1 2^{-\varepsilon n}\right)^2|F|_\sigma,\qquad\text{for all } n\geq 0 \text{ and } F\in\mathcal{F}.$$

From this we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F):}\alpha_{\mathcal{F}}(F')|F'|_\sigma &\leq \sum_{n=0}^{\infty}\sqrt{\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)}\alpha_{\mathcal{F}}(F')^2|F'|_\sigma}\,C_1 2^{-\varepsilon n}\sqrt{|F|_\sigma}\\ &\leq C_1\sqrt{|F|_\sigma}C_\varepsilon\sqrt{\sum_{n=0}^{\infty}2^{-\varepsilon n}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)}\alpha_{\mathcal{F}}(F')^2|F'|_\sigma},\end{aligned}$$

and hence that

$$\begin{aligned}&\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)\left\{\sum_{n=0}^{\infty}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)}\alpha_{\mathcal{F}}(F')|F'|_\sigma\right\}\\ \lesssim\ &\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)\sqrt{|F|_\sigma}\sqrt{\sum_{n=0}^{\infty}2^{-\varepsilon n}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)}\alpha_{\mathcal{F}}(F')^2|F'|_\sigma}\\ \lesssim\ &\left(\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)^2|F|_\sigma\right)^{\frac12}\left(\sum_{n=0}^{\infty}2^{-\varepsilon n}\sum_{F\in\mathcal{F}}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)}\alpha_{\mathcal{F}}(F')^2|F'|_\sigma\right)^{\frac12}\\ \lesssim\ &\|f\|_{L^2(\sigma)}\left(\sum_{F'\in\mathcal{F}}\alpha_{\mathcal{F}}(F')^2|F'|_\sigma\right)^{\frac12}\lesssim\|f\|_{L^2(\sigma)}^2.\end{aligned}$$

This proves (7.6) since $\left\|\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)\mathbf{1}_F\right\|_{L^2(\sigma)}^2$ is dominated by twice the left hand side above.

We will use a construction that permits *iteration* of general corona decompositions.

Lemma 11. *Suppose that* $(C_0,\mathcal{F},\alpha_{\mathcal{F}})$ *constitutes* stopping data *for a function* $f\in L^1_{loc}(\sigma)$*, and that for each* $F\in\mathcal{F}$*,* $\left(C_0,\mathcal{K}(F),\alpha_{\mathcal{K}(F)}\right)$ *constitutes* stopping data *for the corona projection* $\mathsf{P}_{\mathcal{C}_F}^{\sigma}f$*, where in addition* $F\in\mathcal{K}(F)$*. There is a positive constant* C_1*, depending only on* C_0*, such that if*

$$\begin{aligned}\mathcal{K}^*(F) &\equiv \left\{K\in\mathcal{K}(F)\cap\mathcal{C}_F:\alpha_{\mathcal{K}(F)}(K)\geq\alpha_{\mathcal{F}}(F)\right\}\\ \mathcal{K} &\equiv \bigcup_{F\in\mathcal{F}}\mathcal{K}^*(F)\cup\{F\},\\ \alpha_{\mathcal{K}}(K) &\equiv \begin{array}{ccc}\alpha_{\mathcal{K}(F)}(K) & \text{for} & K\in\mathcal{K}^*(F)\setminus\{F\}\\ \max\left\{\alpha_{\mathcal{F}}(F),\alpha_{\mathcal{K}(F)}(F)\right\} & \text{for} & K=F\end{array},\qquad \text{for } F\in\mathcal{F},\end{aligned}$$

the triple $(C_1,\mathcal{K},\alpha_{\mathcal{K}})$ *constitutes* stopping data *for* f. *We refer to the collection of quasicubes* $\mathcal{K}$ *as the* iterated *stopping times, and to the orthogonal decomposition* $f=\sum_{K\in\mathcal{K}}P_{\mathcal{C}_K^{\mathcal{K}}}f$ *as the* iterated *corona decomposition of* f, *where*

$$\mathcal{C}_K^{\mathcal{K}}\equiv\left\{I\in\Omega\mathcal{D}:I\subset K\text{ and }I\not\subset K'\text{ for }K'\prec_{\mathcal{K}}K\right\}.$$

Note that in our definition of $(C_1,\mathcal{K},\alpha_{\mathcal{K}})$ we have 'discarded' from $\mathcal{K}(F)$ all of those $K\in\mathcal{K}(F)$ that are not in the corona $\mathcal{C}_F$, and also all of those $K\in\mathcal{K}(F)$ for which $\alpha_{\mathcal{K}(F)}(K)$ is strictly less than $\alpha_{\mathcal{F}}(F)$. Then the union over F of what remains is our new collection of stopping times. We then define stopping data $\alpha_{\mathcal{K}}(K)$ according to whether or not $K\in\mathcal{F}$: if $K\notin\mathcal{F}$ but $K\in\mathcal{C}_F$ then $\alpha_{\mathcal{K}}(K)$ equals $\alpha_{\mathcal{K}(F)}(K)$, while if $K\in\mathcal{F}$, then $\alpha_{\mathcal{K}}(K)$ is the larger of $\alpha_{\mathcal{K}(F)}(F)$ and $\alpha_{\mathcal{F}}(K)$.

Proof. The monotonicity property (4) for the triple $(C_1,\mathcal{K},\alpha_{\mathcal{K}})$ is obvious from the construction of $\mathcal{K}$ and $\alpha_{\mathcal{K}}(K)$. To establish property (1), we must distinguish between the various coronas $\mathcal{C}_K^{\mathcal{K}}$, $\mathcal{C}_K^{\mathcal{K}(F)}$ and $\mathcal{C}_K^{\mathcal{F}}$ that could be associated with $K\in\mathcal{K}$, when K belongs to any of the stopping trees $\mathcal{K}$, $\mathcal{K}(F)$ or $\mathcal{F}$. Suppose now that $I\in\mathcal{C}_K^{\mathcal{K}}$ for some $K\in\mathcal{K}$. Then there is a unique $F\in\mathcal{F}$ such that $\mathcal{C}_K^{\mathcal{K}}\subset\mathcal{C}_K^{\mathcal{K}(F)}\subset\mathcal{C}_F^{\mathcal{F}}$, and so $\mathbb{E}_I^{\sigma}|f|\leq\alpha_{\mathcal{F}}(F)$ by property (1) for the triple $(C_0,\mathcal{F},\alpha_{\mathcal{F}})$. Then $\alpha_{\mathcal{F}}(F)\leq\alpha_{\mathcal{K}}(K)$ follows from the definition of $\alpha_{\mathcal{K}}(K)$, and we have property (1) for the triple $(C_1,\mathcal{K},\alpha_{\mathcal{K}})$. Property (2) holds for the triple $(C_1,\mathcal{K},\alpha_{\mathcal{K}})$ since if $K\in\mathcal{C}_F^{\mathcal{F}}$, then

$$\begin{aligned}\sum_{K'\preceq_{\mathcal{K}}K}|K'|_{\sigma} &= \sum_{K'\in\mathcal{K}(F):\ K'\subset K}|K'|_{\sigma}+\sum_{F'\prec_{\mathcal{F}}F:\ F'\subset K}\ \sum_{K'\in\mathcal{K}(F')}|K'|_{\sigma}\\ &\leq C_0|K|_{\sigma}+\sum_{F'\prec_{\mathcal{F}}F:\ F'\subset K}C_0|F'|_{\sigma}\leq 2C_0^2|K|_{\sigma}.\end{aligned}$$

Finally, property (3) holds for the triple $(C_1,\mathcal{K},\alpha_{\mathcal{K}})$ since

$$\begin{aligned}\sum_{K\in\mathcal{K}}\alpha_{\mathcal{K}}(K)^2|K|_{\sigma} &\leq \sum_{F\in\mathcal{F}}\sum_{K\in\mathcal{K}(F)}\alpha_{\mathcal{K}(F)}(K)^2|K|_{\sigma}+\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)^2|F|_{\sigma}\\ &\leq \sum_{F\in\mathcal{F}}C_0^2\left\|\mathsf{P}_{\mathcal{C}_F}^{\sigma}f\right\|_{L^2(\sigma)}^2+C_0^2\|f\|_{L^2(\sigma)}^2\leq 2C_0^2\|f\|_{L^2(\sigma)}^2.\end{aligned}$$

$\square$

7.4. **Doubly iterated coronas and the NTV quasicube size splitting.** Here is a brief schematic diagram of the decompositions, with bounds in $\square$, used in this subsection:

$$\begin{array}{ccccccc}
\left\langle T_{\sigma}^{\alpha}f,g\right\rangle_{\omega} & & & & & & \\
\downarrow & & & & & & \\
\mathsf{B}_{\Subset_{\boldsymbol{\rho},\varepsilon}}(f,g) & + & \mathsf{B}_{_{\boldsymbol{\rho},\varepsilon}\Supset}(f,g) & + & \mathsf{B}_{\cap}(f,g) & + & \mathsf{B}_{\diagup}(f,g)\\
\downarrow & & \boxed{\text{dual}} & & \boxed{\mathcal{NTV}_{\alpha}} & & \boxed{\mathcal{NTV}_{\alpha}}\\
\downarrow & & & & & & \\
\mathsf{T}_{\text{diagonal}}(f,g) & + & \mathsf{T}_{\text{far below}}(f,g) & + & \mathsf{T}_{\text{far above}}(f,g) & + & \mathsf{T}_{\text{disjoint}}(f,g)\\
\downarrow & & \downarrow & & \boxed{\emptyset} & & \boxed{\emptyset}\\
\downarrow & & \downarrow & & & & \\
\mathsf{B}_{\Subset_{\boldsymbol{\rho},\varepsilon}}^{A}(f,g) & & \mathsf{T}_{\text{far below}}^{1}(f,g) & + & \mathsf{T}_{\text{far below}}^{2}(f,g) & & \\
\downarrow & & \boxed{\mathcal{NTV}_{\alpha}+\mathcal{E}_{\alpha}} & & \boxed{\mathcal{NTV}_{\alpha}} & & \\
\downarrow & & & & & & \\
\mathsf{B}_{stop}^{A}(f,g) & + & \mathsf{B}_{paraproduct}^{A}(f,g) & + & \mathsf{B}_{neighbour}^{A}(f,g) & & \\
\boxed{\mathcal{E}_{\alpha}^{\text{deep}}+\sqrt{A_2^{\alpha}}} & & \boxed{\mathfrak{T}_{T^{\alpha}}} & & \boxed{\sqrt{A_2^{\alpha}}} & &
\end{array}$$

We begin with the NTV *quasicube size splitting* of the inner product $\left\langle T_{\sigma}^{\alpha}f,g\right\rangle_{\omega}$ - and later apply the iterated corona construction - that splits the pairs of quasicubes (I,J) in a simultaneous quasiHaar decomposition of f and g into four groups, namely those pairs that:

(1) are below the size diagonal and $\boldsymbol{\rho}$-deeply embedded,
(2) are above the size diagonal and $\boldsymbol{\rho}$-deeply embedded,

(3) are disjoint, and
(4) are of $\boldsymbol{\rho}$-comparable size.

More precisely we have

$$\begin{aligned}
\left\langle T_{\sigma}^{\alpha}f,g\right\rangle_{\omega} &= \sum_{I\in\Omega\mathcal{D}^{\sigma},\ J\in\Omega\mathcal{D}^{\omega}} \left\langle T_{\sigma}^{\alpha}\left(\triangle_{I}^{\sigma}f\right),\left(\triangle_{I}^{\omega}g\right)\right\rangle_{\omega} \\
&= \sum_{\substack{I\in\Omega\mathcal{D}^{\sigma},\ J\in\Omega\mathcal{D}^{\omega}\\ J\Subset_{\boldsymbol{\rho},\varepsilon}I}} \left\langle T_{\sigma}^{\alpha}\left(\triangle_{I}^{\sigma}f\right),\left(\triangle_{J}^{\omega}g\right)\right\rangle_{\omega} + \sum_{\substack{I\in\Omega\mathcal{D}^{\sigma},\ J\in\Omega\mathcal{D}^{\omega}\\ J_{\boldsymbol{\rho},\varepsilon}\Supset I}} \left\langle T_{\sigma}^{\alpha}\left(\triangle_{I}^{\sigma}f\right),\left(\triangle_{J}^{\omega}g\right)\right\rangle_{\omega} \\
&\quad + \sum_{\substack{I\in\Omega\mathcal{D}^{\sigma},\ J\in\Omega\mathcal{D}^{\omega}\\ J\cap I=\emptyset}} \left\langle T_{\sigma}^{\alpha}\left(\triangle_{I}^{\sigma}f\right),\left(\triangle_{J}^{\omega}g\right)\right\rangle_{\omega} + \sum_{\substack{I\in\Omega\mathcal{D}^{\sigma},\ J\in\Omega\mathcal{D}^{\omega}\\ 2^{-\boldsymbol{\rho}}\leq\ell(J)/\ell(I)\leq2^{\boldsymbol{\rho}}}} \left\langle T_{\sigma}^{\alpha}\left(\triangle_{I}^{\sigma}f\right),\left(\triangle_{J}^{\omega}g\right)\right\rangle_{\omega} \\
&= \mathsf{B}_{\Subset_{\boldsymbol{\rho},\varepsilon}}\left(f,g\right)+\mathsf{B}_{\boldsymbol{\rho},\varepsilon\Supset}\left(f,g\right)+\mathsf{B}_{\cap}\left(f,g\right)+\mathsf{B}_{\diagup}\left(f,g\right).
\end{aligned}$$

Lemma 9 in the section on NTV peliminaries show that the *disjoint* and *comparable* forms $\mathsf{B}_{\cap}\left(f,g\right)$ and $\mathsf{B}_{\diagup}\left(f,g\right)$ are both bounded by the $\mathcal{A}_{2}^{\alpha}$, $A_{2}^{\alpha,\mathrm{punct}}$, quasitesting and quasiweak boundedness property constants. The *below* and *above* forms are clearly symmetric, so we need only consider the form $\mathsf{B}_{\Subset_{\boldsymbol{\rho},\varepsilon}}\left(f,g\right)$, to which we turn for the remainder of the proof.

In order to bound the below form $\mathsf{B}_{\Subset_{\boldsymbol{\rho},\varepsilon}}\left(f,g\right)$, we will apply two different corona decompositions in succession to the function $f\in L^{2}\left(\sigma\right)$, gaining structure with each application; first to a boundedness property for f, and then to a regularizing property of the weight σ. We first apply the Calderón-Zygmund corona decomposition to the function $f\in L^{2}\left(\sigma\right)$ obtain

$$f=\sum_{F\in\mathcal{F}}\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f.$$

Then for each fixed $F\in\mathcal{F}$, construct the *quasienergy* corona decomposition $\left\{\mathcal{C}_{S}^{\sigma}\right\}_{S\in\mathcal{S}(F)}$ corresponding to the weight pair $\left(\sigma,\omega\right)$ with top quasicube $S_{0}=F$, as given in Definition 9. At this point we apply Lemma 11 to obtain iterated stopping times $\mathcal{S}$ and iterated stopping data $\left\{\alpha_{\mathcal{S}}\left(S\right)\right\}_{S\in\mathcal{S}}$. This gives us the following *double corona decomposition* of f,

$$f=\sum_{F\in\mathcal{F}}\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f=\sum_{F\in\mathcal{F}}\sum_{S\in\mathcal{S}^{\ast}(F)\cup\{F\}}\mathsf{P}_{\mathcal{C}_{S}^{\sigma}}^{\sigma}\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f=\sum_{S\in\mathcal{S}}\mathsf{P}_{\mathcal{C}_{S}^{\sigma}}^{\sigma}f\equiv\sum_{A\in\mathcal{A}}\mathsf{P}_{\mathcal{C}_{A}}^{\sigma}f, \tag{7.7}$$

where $\mathcal{A}\equiv\mathcal{S}$ is the double stopping collection for f. We are relabeling the double corona as $\mathcal{A}$ here so as to minimize confusion. We now record the main facts proved above for the double corona.

Lemma 12. *The data $\mathcal{A}$ and $\left\{\alpha_{\mathcal{A}}\left(A\right)\right\}_{A\in\mathcal{A}}$ satisfy properties (1), (2), (3) and (4) in Definition 10.*

To bound $\mathsf{B}_{\Subset_{\boldsymbol{\rho},\varepsilon}}\left(f,g\right)$ we fix the stopping data $\mathcal{A}$ and $\left\{\alpha_{\mathcal{A}}\left(A\right)\right\}_{A\in\mathcal{A}}$ constructed above with the double iterated corona. We now consider the following *canonical splitting* of the form $\mathsf{B}_{\Subset_{\boldsymbol{\rho},\varepsilon}}\left(f,g\right)$ that involves the quasiHaar corona projections $\mathsf{P}_{\mathcal{C}_{A}}^{\sigma}$ acting on f and the $\boldsymbol{\tau}$*-shifted* quasiHaar corona projections $\mathsf{P}_{\mathcal{C}_{B}^{\boldsymbol{\tau}-\mathrm{shift}}}^{\omega}$ acting on g. Here the $\boldsymbol{\tau}$-shifted corona $\mathcal{C}_{B}^{\boldsymbol{\tau}-\mathrm{shift}}$ is defined to include only those quasicubes $J\in\mathcal{C}_{B}$ that are *not* $\boldsymbol{\tau}$-nearby B, and to include also such quasicubes J which in addition *are* $\boldsymbol{\tau}$-nearby in the children B' of B.

Definition 12. *The parameters $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ are now fixed to satisfy*

$$\boldsymbol{\tau}>\mathbf{r}\text{ and }\boldsymbol{\rho}>\mathbf{r}+\boldsymbol{\tau},$$

where $\mathbf{r}$ is the goodness parameter already fixed.

Definition 13. *For $B\in\mathcal{A}$ we define*

$$\mathcal{C}_{B}^{\boldsymbol{\tau}-\mathrm{shift}}=\left\{J\in\mathcal{C}_{B}:J\Subset_{\boldsymbol{\tau},\varepsilon}B\right\}\cup\bigcup_{B'\in\mathfrak{C}_{\mathcal{A}}(B)}\left\{J\in\Omega\mathcal{D}:J\Subset_{\boldsymbol{\tau},\varepsilon}B\text{ and }J\text{ is }\boldsymbol{\tau}\text{-nearby in }B'\right\}.$$

We will use repeatedly the fact that the $\boldsymbol{\tau}$-shifted coronas $\mathcal{C}_{B}^{\boldsymbol{\tau}-\mathrm{shift}}$ have overlap bounded by $\boldsymbol{\tau}$:

$$\sum_{B\in\mathcal{A}}\mathbf{1}_{\mathcal{C}_{B}^{\boldsymbol{\tau}-\mathrm{shift}}}\left(J\right)\leq\boldsymbol{\tau},\qquad J\in\Omega\mathcal{D}. \tag{7.8}$$

The forms $\mathsf{B}_{\Subset_{\boldsymbol{\rho},\varepsilon}}(f,g)$ are no longer linear in f and g as the 'cut' is determined by the coronas $\mathcal{C}_F$ and $\mathcal{C}_G^{\boldsymbol{\tau}-\mathrm{shift}}$, which depend on f as well as the measures σ and ω. However, if the coronas are held fixed, then the forms can be considered bilinear in f and g. It is convenient at this point to introduce the following shorthand notation:

$$\left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^\sigma f\right),\mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}^\omega g\right\rangle_\omega^{\Subset_{\boldsymbol{\rho},\varepsilon}}\equiv\sum_{\substack{I\in\mathcal{C}_A\text{ and }J\in\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}\\ J\Subset_{\boldsymbol{\rho},\varepsilon}I}}\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega .$$

We then have the canonical splitting,

$$\begin{aligned}
&\mathsf{B}_{\Subset_{\boldsymbol{\rho},\varepsilon}}(f,g) \qquad (7.9)\\
=&\sum_{A,B\in\mathcal{A}}\left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^\sigma f\right),\mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}^\omega g\right\rangle_\omega^{\Subset_{\boldsymbol{\rho},\varepsilon}}\\
=&\sum_{A\in\mathcal{A}}\left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^\sigma f\right),\mathsf{P}_{\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}}^\omega g\right\rangle_\omega^{\Subset_{\boldsymbol{\rho},\varepsilon}}+\sum_{\substack{A,B\in\mathcal{A}\\ B\subsetneqq A}}\left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^\sigma f\right),\mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}^\omega g\right\rangle_\omega^{\Subset_{\boldsymbol{\rho},\varepsilon}}\\
&+\sum_{\substack{A,B\in\mathcal{A}\\ B\supsetneqq A}}\left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^\sigma f\right),\mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}^\omega g\right\rangle_\omega^{\Subset_{\boldsymbol{\rho},\varepsilon}}+\sum_{\substack{A,B\in\mathcal{A}\\ A\cap B=\emptyset}}\left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^\sigma f\right),\mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}^\omega g\right\rangle_\omega^{\Subset_{\boldsymbol{\rho},\varepsilon}}\\
\equiv&\ \mathsf{T}_{\mathrm{diagonal}}(f,g)+\mathsf{T}_{\mathrm{far\,below}}(f,g)+\mathsf{T}_{\mathrm{far\,above}}(f,g)+\mathsf{T}_{\mathrm{disjoint}}(f,g).
\end{aligned}$$

Now the final two terms $\mathsf{T}_{\mathrm{far\,above}}(f,g)$ and $\mathsf{T}_{\mathrm{disjoint}}(f,g)$ each vanish since there are no pairs $(I,J)\in\mathcal{C}_A\times\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}$ with both (**i**) $J\Subset_{\boldsymbol{\rho},\varepsilon}I$ and (**ii**) either $B\subsetneqq A$ or $B\cap A=\emptyset$.

The *far below* term $\mathsf{T}_{\mathrm{far\,below}}(f,g)$ is bounded using the Intertwining Proposition and the control of functional energy condition by the energy condition given in the next two sections. Indeed, assuming these two results, we have from $\boldsymbol{\tau}<\boldsymbol{\rho}$ that

$$\begin{aligned}
\mathsf{T}_{\mathrm{far\,below}}(f,g)&=\sum_{\substack{A,B\in\mathcal{A}\\ B\subsetneqq A}}\ \sum_{\substack{I\in\mathcal{C}_A\text{ and }J\in\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}\\ J\Subset_{\boldsymbol{\rho},\varepsilon}I}}\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega\\
&=\sum_{B\in\mathcal{A}}\ \sum_{A\in\mathcal{A}:\ B\subsetneqq A}\ \sum_{\substack{I\in\mathcal{C}_A\text{ and }J\in\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}\\ J\Subset_{\boldsymbol{\rho},\varepsilon}I}}\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega\\
&=\sum_{B\in\mathcal{A}}\ \sum_{A\in\mathcal{A}:\ B\subsetneqq A}\ \sum_{I\in\mathcal{C}_A\text{ and }J\in\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega\\
&\quad-\sum_{B\in\mathcal{A}}\ \sum_{A\in\mathcal{A}:\ B\subsetneqq A}\ \sum_{\substack{I\in\mathcal{C}_A\text{ and }J\in\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}\\ J\not\Subset_{\boldsymbol{\rho},\varepsilon}I}}\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega\\
&=\mathsf{T}_{\mathrm{far\,below}}^1(f,g)-\mathsf{T}_{\mathrm{far\,below}}^2(f,g).
\end{aligned}$$

Now $\mathsf{T}_{\mathrm{far\ \ below}}^2(f,g)$ is bounded by $\mathcal{NTV}_\alpha$ by Lemma 9 since J is good if $\triangle_J^\omega g\neq 0$.

The form $\mathsf{T}_{\mathrm{far\,below}}^1(f,g)$ can be written as

$$\begin{aligned}
\mathsf{T}_{\mathrm{far\,below}}^1(f,g)&=\sum_{B\in\mathcal{A}}\ \sum_{I\in\Omega\mathcal{D}:\ B\subsetneqq I}\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),g_B\right\rangle_\omega ;\\
\text{where }g_B&\equiv\sum_{J\in\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}\triangle_J^\omega g .
\end{aligned}$$

The Intertwining Proposition 1 applies to this latter form and shows that it is bounded by $\mathcal{NTV}_\alpha+\mathfrak{F}_\alpha$. Then Proposition 2 shows that $\mathfrak{F}_\alpha\lesssim\mathcal{NTV}_\alpha+\mathcal{E}_\alpha$, which completes the proof that

$$\left|\mathsf{T}_{\mathrm{far\,below}}(f,g)\right|\lesssim\left(\mathcal{NTV}_\alpha+\mathcal{E}_\alpha\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)} . \tag{7.10}$$

The boundedness of the diagonal term $\mathsf{T}_{\mathrm{diagonal}}(f,g)$ will then be reduced to the forms in the paraproduct/neighbour/stopping form decomposition of NTV. The stopping form is then further split into two

sublinear forms in (10.6) below, where the boundedness of the more difficult of the two is treated by adapting the stopping time and recursion of M. Lacey [Lac]. More precisely, to handle the diagonal term $\mathsf{T}_{\mathrm{diagonal}}(f,g)$, it is enough to consider the individual corona pieces

$$\mathsf{B}_{\Subset_{\rho,\varepsilon}}^{A}(f,g)\equiv\left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^\sigma f\right),\mathsf{P}_{\mathcal{C}_A^{\tau-\mathrm{shift}}}^\omega g\right\rangle_\omega^{\Subset},$$

and to prove the following estimate:

$$\left|\mathsf{B}_{\Subset_{\rho,\varepsilon}}^{A}(f,g)\right|\lesssim\left(\mathcal{NTV}_\alpha+\mathcal{E}_\alpha\right)\left(\alpha_{\mathcal{A}}(A)\sqrt{|A|_\sigma}+\left\|\mathsf{P}_{\mathcal{C}_A}^\sigma f\right\|_{L^2(\sigma)}\right)\left\|\mathsf{P}_{\mathcal{C}_A^{\tau-\mathrm{shift}}}^\omega g\right\|_{L^2(\omega)}.$$

Indeed, we then have from Cauchy-Schwarz that

$$\begin{aligned}
&\sum_{A\in\mathcal{A}}\left|\mathsf{B}_{\Subset_{\rho,\varepsilon}}^{A}(f,g)\right|=\sum_{A\in\mathcal{A}}\left|\mathsf{B}_{\Subset_{\rho,\varepsilon}}^{A}\left(\mathsf{P}_{\mathcal{C}_A}^\sigma f,\mathsf{P}_{\mathcal{C}_A^{\tau-\mathrm{shift}}}^\omega g\right)\right|\\
\lesssim&\ \left(\mathcal{NTV}_\alpha+\mathcal{E}_\alpha\right)\left(\sum_{A\in\mathcal{A}}\alpha_{\mathcal{A}}(A)^2|A|_\sigma+\left\|\mathsf{P}_{\mathcal{C}_A}^\sigma f\right\|_{L^2(\sigma)}^2\right)^{\frac12}\left(\sum_{A\in\mathcal{A}}\left\|\mathsf{P}_{\mathcal{C}_A^{\tau-\mathrm{shift}}}^\omega g\right\|_{L^2(\omega)}^2\right)^{\frac12}\\
\lesssim&\ \left(\mathcal{NTV}_\alpha+\mathcal{E}_\alpha\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)},
\end{aligned}$$

where the last line uses 'quasi' orthogonality in f and orthogonality in both f and g.

Following arguments in [NTV4], [Vol] and [LaSaShUr], we now use the paraproduct / neighbour / stopping splitting of NTV to reduce boundedness of $\mathsf{B}_{\Subset_{\rho,\varepsilon}}^{A}(f,g)$ to boundedness of the associated stopping form

$$\mathsf{B}_{stop}^{A}(f,g)\equiv\sum_{I\in\operatorname{supp}\widehat{f}}\ \sum_{J:\ J\Subset_{\rho,\varepsilon}I\text{ and }I_J\notin\mathcal{A}}\left(\mathbb{E}_{I_J}^\sigma\bigtriangleup_I^\sigma f\right)\left\langle T_\sigma^\alpha\mathbf{1}_{A\setminus I_J},\bigtriangleup_J^\omega g\right\rangle_\omega,\tag{7.11}$$

where f is supported in the quasicube A and its expectations $\mathbb{E}_I^\sigma|f|$ are bounded by $\alpha_{\mathcal{A}}(A)$ for $I\in\mathcal{C}_A^\sigma$, the quasiHaar support of f is contained in the corona $\mathcal{C}_A^\sigma$, and the quasiHaar support of g is contained in $\mathcal{C}_A^{\tau-\mathrm{shift}}$, and where I_J is the $\Omega\mathcal{D}$-child of I that contains J. Indeed, to see this, we note that $\bigtriangleup_I^\sigma f=\mathbf{1}_I\bigtriangleup_I^\sigma f$ and write both

$$\begin{aligned}
\mathbf{1}_I&=\mathbf{1}_{I_J}+\sum_{\theta(I_J)\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\setminus\{I_J\}}\mathbf{1}_{\theta(I_J)},\\
\mathbf{1}_{I_J}&=\mathbf{1}_A-\mathbf{1}_{A\setminus I_J},
\end{aligned}$$

where $\theta(I_J)\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\setminus\{I_J\}$ ranges over the 2^n-1 $\Omega\mathcal{D}$-children of I other than the child I_J that contains J. Then we obtain

$$\begin{aligned}
\left\langle T_\sigma^\alpha\bigtriangleup_I^\sigma f,\bigtriangleup_J^\omega g\right\rangle_\omega&=\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_J}\bigtriangleup_I^\sigma f\right),\bigtriangleup_J^\omega g\right\rangle_\omega+\sum_{\theta(I_J)\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\setminus\{I_J\}}\left\langle T_\sigma^\alpha\left(\mathbf{1}_{\theta(I_J)}\bigtriangleup_I^\sigma f\right),\bigtriangleup_J^\omega g\right\rangle_\omega\\
&=\left(\mathbb{E}_{I_J}^\sigma\bigtriangleup_I^\sigma f\right)\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_J}\right),\bigtriangleup_J^\omega g\right\rangle_\omega+\sum_{\theta(I_J)\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\setminus\{I_J\}}\left\langle T_\sigma^\alpha\left(\mathbf{1}_{\theta(I_J)}\bigtriangleup_I^\sigma f\right),\bigtriangleup_J^\omega g\right\rangle_\omega\\
&=\left(\mathbb{E}_{I_J}^\sigma\bigtriangleup_I^\sigma f\right)\left\langle T_\sigma^\alpha\mathbf{1}_A,\bigtriangleup_J^\omega g\right\rangle_\omega\\
&\quad-\left(\mathbb{E}_{I_J}^\sigma\bigtriangleup_I^\sigma f\right)\left\langle T_\sigma^\alpha\mathbf{1}_{A\setminus I_J},\bigtriangleup_J^\omega g\right\rangle_\omega\\
&\quad+\sum_{\theta(I_J)\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\setminus\{I_J\}}\left\langle T_\sigma^\alpha\left(\mathbf{1}_{\theta(I_J)}\bigtriangleup_I^\sigma f\right),\bigtriangleup_J^\omega g\right\rangle_\omega,
\end{aligned}$$

and the corresponding NTV splitting of $\mathsf{B}^A_{\Subset_{\boldsymbol{\rho},\varepsilon}}(f,g)$:

$$\begin{aligned}
\mathsf{B}^A_{\Subset_{\boldsymbol{\rho},\varepsilon}}(f,g) &= \left\langle T^\alpha_\sigma\left(\mathsf{P}^\sigma_{\mathcal{C}_A}f\right),\mathsf{P}^\omega_{\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}}g\right\rangle_\omega^{\Subset_{\boldsymbol{\rho},\varepsilon}} = \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}\\ J\Subset_{\boldsymbol{\rho},\varepsilon}I}} \left\langle T^\alpha_\sigma\left(\triangle^\sigma_I f\right),\triangle^\omega_J g\right\rangle_\omega \\
&= \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}\\ J\Subset_{\boldsymbol{\rho},\varepsilon}I}} \left(\mathbb{E}^\sigma_{I_J}\triangle^\sigma_I f\right)\left\langle T^\alpha_\sigma\mathbf{1}_A,\triangle^\omega_J g\right\rangle_\omega \\
&\quad - \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}\\ J\Subset_{\boldsymbol{\rho},\varepsilon}I}} \left(\mathbb{E}^\sigma_{I_J}\triangle^\sigma_I f\right)\left\langle T^\alpha_\sigma\mathbf{1}_{A\setminus I_J},\triangle^\omega_J g\right\rangle_\omega \\
&\quad + \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}\\ J\Subset_{\boldsymbol{\rho},\varepsilon}I}} \sum_{\theta(I_J)\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\setminus\{I_J\}} \left\langle T^\alpha_\sigma\left(\mathbf{1}_{\theta(I_J)}\triangle^\sigma_I f\right),\triangle^\omega_J g\right\rangle_\omega \\
&\equiv \mathsf{B}^A_{paraproduct}(f,g) - \mathsf{B}^A_{stop}(f,g) + \mathsf{B}^A_{neighbour}(f,g).
\end{aligned}$$

The paraproduct form $\mathsf{B}^A_{paraproduct}(f,g)$ is easily controlled by the testing condition for T^α. Indeed, we have

$$\begin{aligned}
\mathsf{B}^A_{paraproduct}(f,g) &= \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}\\ J\Subset_{\boldsymbol{\rho},\varepsilon}I}} \left(\mathbb{E}^\sigma_{I_J}\triangle^\sigma_I f\right)\left\langle T^\alpha_\sigma\mathbf{1}_A,\triangle^\omega_J g\right\rangle_\omega \\
&= \sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}} \left\langle T^\alpha_\sigma\mathbf{1}_A,\triangle^\omega_J g\right\rangle_\omega \left\{\sum_{I\in\mathcal{C}_A:\ J\Subset_{\boldsymbol{\rho},\varepsilon}I}\left(\mathbb{E}^\sigma_{I_J}\triangle^\sigma_I f\right)\right\} \\
&= \sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}} \left\langle T^\alpha_\sigma\mathbf{1}_A,\triangle^\omega_J g\right\rangle_\omega \left\{\mathbb{E}^\sigma_{I^\natural(J)_J}f-\mathbb{E}^\sigma_A f\right\} \\
&= \left\langle T^\alpha_\sigma\mathbf{1}_A,\sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}}\left\{\mathbb{E}^\sigma_{I^\natural(J)_J}f-\mathbb{E}^\sigma_A f\right\}\triangle^\omega_J g\right\rangle_\omega ,
\end{aligned}$$

where $I^\natural(J)$ denotes the smallest quasicube $I\in\mathcal{C}_A$ such that $J\Subset_{\boldsymbol{\rho},\varepsilon}I$, and of course $I^\natural(J)_J$ denotes its child containing J. We claim that by construction of the corona we have $I^\natural(J)_J\notin\mathcal{A}$, and so $\left|\mathbb{E}^\sigma_{I^\natural(J)_J}f\right|\lesssim\mathbb{E}^\sigma_A|f|\leq\alpha_{\mathcal{A}}(A)$. Indeed, in our application of the stopping form we have $f=\mathsf{P}^\sigma_{\mathcal{C}_A}f$ and $g=\mathsf{P}^\omega_{\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}}g$, and the definitions of the coronas $\mathcal{C}_A$ and $\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}$ together with $\mathbf{r}<\boldsymbol{\tau}<\boldsymbol{\rho}$ imply that $I^\natural(J)_J\notin\mathcal{A}$ for $J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}$.

Thus from the orthogonality of the quasiHaar projections $\triangle^\omega_J g$ and the bound on the coefficients $\left|\mathbb{E}^\sigma_{I^\natural(J)_J}f-\mathbb{E}^\sigma_A f\right|\lesssim\alpha_{\mathcal{A}}(A)$ we have

$$\begin{aligned}
\left|\mathsf{B}^A_{paraproduct}(f,g)\right| &= \left|\left\langle T^\alpha_\sigma\mathbf{1}_A,\sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}}\left\{\mathbb{E}^\sigma_{I^\natural(J)_J}f-\mathbb{E}^\sigma_A f\right\}\triangle^\omega_J g\right\rangle_\omega\right| \\
&\lesssim \alpha_{\mathcal{A}}(A)\ \left\|\mathbf{1}_A T^\alpha_\sigma\mathbf{1}_A\right\|_{L^2(\omega)}\ \left\|\mathsf{P}^\omega_{\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}}g\right\|_{L^2(\omega)} \\
&\leq \mathfrak{T}_{T^\alpha}\ \alpha_{\mathcal{A}}(A)\ \sqrt{|A|_\sigma}\ \left\|\mathsf{P}^\omega_{\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}}g\right\|_{L^2(\omega)},
\end{aligned}$$

because $\left\|\sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}}\lambda_J\triangle^\omega_J g\right\|_{L^2(\omega)}\leq\left(\sup_J|\lambda_J|\right)\left\|\sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}}\triangle^\omega_J g\right\|_{L^2(\omega)}$.

Next, the neighbour form $\mathsf{B}^A_{neighbour}(f,g)$ is easily controlled by the A^α_2 condition using the Energy Lemma 8 and the fact that the quasicubes J are good. In particular, the information encoded in the stopping tree

$\mathcal{A}$ plays no role here. We have

$$\mathsf{B}_{neighbour}^{A}(f,g)=\sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon}I}}\sum_{\theta(I_J)\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\setminus\{I_J\}}\left\langle T_\sigma^\alpha\left(\mathbf{1}_{\theta(I_J)}\bigtriangleup_I^\sigma f\right),\bigtriangleup_J^\omega g\right\rangle_\omega .$$

Recall that I_J is the child of I that contains J. Fix $\theta(I_J)\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\setminus\{I_J\}$ momentarily, and an integer $s\geq\mathbf{r}$. The inner product to be estimated is

$$\left\langle T_\sigma^\alpha(\mathbf{1}_{\theta(I_J)}\sigma\Delta_I^\sigma f),\Delta_J^\omega g\right\rangle_\omega ,$$

i.e.

$$\left\langle T_\sigma^\alpha\left(\mathbf{1}_{\theta(I_J)}\bigtriangleup_I^\sigma f\right),\bigtriangleup_J^\omega g\right\rangle_\omega=\mathbb{E}_{\theta(I_J)}^\sigma\Delta_I^\sigma f\cdot\left\langle T_\sigma^\alpha\left(\mathbf{1}_{\theta(I_J)}\right),\bigtriangleup_J^\omega g\right\rangle_\omega .$$

Thus we can write

$$\mathsf{B}_{neighbour}^{A}(f,g)=\sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon}I}}\sum_{\theta(I_J)\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\setminus\{I_J\}}\left(\mathbb{E}_{\theta(I_J)}^\sigma\Delta_I^\sigma f\right)\left\langle T_\sigma^\alpha\left(\mathbf{1}_{\theta(I_J)}\sigma\right),\Delta_J^\omega g\right\rangle_\omega \tag{7.12}$$

Now we will use the following fractional analogue of the Poisson inequality in [Vol]. We remind the reader that there are absolute positive constants c,C such that $c\left|J\right|^{\frac{1}{n}}\leq\ell(J)\leq C\left|J\right|^{\frac{1}{n}}$ for all quasicubes J, and that we defined the quasidistance $\operatorname{qdist}(E,F)$ between two sets E and F to be the Euclidean distance $\operatorname{dist}\left(\Omega^{-1}E,\Omega^{-1}F\right)$ between the preimages under Ω of the sets E and F.

Lemma 13. *Suppose that* $J\subset I\subset K$ *and that* $\operatorname{qdist}(J,\partial I)>\frac{1}{2}\ell(J)^\varepsilon\ell(I)^{1-\varepsilon}$. *Then*

$$\mathrm{P}^\alpha(J,\sigma\mathbf{1}_{K\setminus I})\lesssim\left(\frac{\ell(J)}{\ell(I)}\right)^{1-\varepsilon(n+1-\alpha)}\mathrm{P}^\alpha(I,\sigma\mathbf{1}_{K\setminus I}). \tag{7.13}$$

Proof. We have

$$\mathrm{P}^\alpha\left(J,\sigma\chi_{K\setminus I}\right)\approx\sum_{k=0}^\infty 2^{-k}\frac{1}{\left|2^kJ\right|^{1-\frac{\alpha}{n}}}\int_{(2^kJ)\cap(K\setminus I)}d\sigma,$$

and $\left(2^kJ\right)\cap(K\setminus I)\neq\emptyset$ requires

$$\operatorname{qdist}(J,K\setminus I)\leq c2^k\ell(J),$$

for some dimensional constant $c>0$. Let k_0 be the smallest such k. By our distance assumption we must then have

$$\tfrac{1}{2}\ell(J)^\varepsilon\ell(I)^{1-\varepsilon}\leq\operatorname{qdist}(J,\partial I)\leq c2^{k_0}\ell(J),$$

or

$$2^{-k_0-1}\leq c\left(\frac{\ell(J)}{\ell(I)}\right)^{1-\varepsilon}.$$

Now let k_1 be defined by $2^{k_1}\equiv\frac{\ell(I)}{\ell(J)}$. Then assuming $k_1>k_0$ (the case $k_1\leq k_0$ is similar) we have

$$\begin{aligned}
\mathrm{P}^\alpha\left(J,\sigma\chi_{K\setminus I}\right) &\approx \left\{\sum_{k=k_0}^{k_1}+\sum_{k=k_1}^{\infty}\right\}2^{-k}\frac{1}{\left|2^kJ\right|^{1-\frac{\alpha}{n}}}\int_{(2^kJ)\cap(K\setminus I)}d\sigma\\
&\lesssim 2^{-k_0}\frac{\left|I\right|^{1-\frac{\alpha}{n}}}{\left|2^{k_0}J\right|^{1-\frac{\alpha}{n}}}\left(\frac{1}{\left|I\right|^{1-\frac{\alpha}{n}}}\int_{\left(2^{k_1}J\right)\cap(K\setminus I)}d\sigma\right)+2^{-k_1}\mathrm{P}^\alpha\left(I,\sigma\chi_{\setminus I}\right)\\
&\lesssim \left(\frac{\ell(J)}{\ell(I)}\right)^{(1-\varepsilon)(n+1-\alpha)}\left(\frac{\ell(I)}{\ell(J)}\right)^{n-\alpha}\mathrm{P}^\alpha\left(I,\sigma\chi_{K\setminus I}\right)+\frac{\ell(J)}{\ell(I)}\mathrm{P}^\alpha\left(I,\sigma\chi_{K\setminus I}\right),
\end{aligned}$$

which is the inequality (7.13). $\square$

Now fix $I_0,I_\theta\in\mathfrak{C}_{\Omega\mathcal{D}}(I)$ with $I_0\neq I_\theta$ and assume that $J\Subset_{\mathbf{r},\varepsilon}I_0$. Let $\frac{\ell(J)}{\ell(I_0)}=2^{-s}$ in the pivotal estimate in the Energy Lemma 8 with $J\subset I_0\subset I$ to obtain

$$\begin{aligned}
\left|\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\sigma\right),\Delta_J^\omega g\rangle_\omega\right| &\lesssim \left\|\Delta_J^\omega g\right\|_{L^2(\omega)}\sqrt{\left|J\right|_\omega}\mathrm{P}^\alpha\left(J,\mathbf{1}_{I_\theta}\sigma\right)\\
&\lesssim \left\|\Delta_J^\omega g\right\|_{L^2(\omega)}\sqrt{\left|J\right|_\omega}\cdot 2^{-(1-\varepsilon(n+1-\alpha))s}\mathrm{P}^\alpha\left(I_0,\mathbf{1}_{I_\theta}\sigma\right).
\end{aligned}$$

Here we are using (7.13), which applies since $J \subset I_0$.

In the sum below, we keep the side lengths of the quasicubes J fixed at 2^{-s} times that of I, and of course take $J \subset I_0$. We estimate

$$\begin{aligned}
A(I,I_0,I_\theta,s) &\equiv \sum_{J\,:\,2^s\ell(J)=\ell(I):J\subset I_0} \left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\sigma\Delta_I^\sigma f\right),\Delta_J^\omega g\right\rangle_\omega\right| \\
&\leq 2^{-(1-\varepsilon(n+1-\alpha))s}\left|\mathbb{E}_{I_\theta}^\sigma\Delta_I^\sigma f\right|\ \mathrm{P}^\alpha(I_0,\mathbf{1}_{I_\theta}\sigma)\sum_{J\,:\,2^s\ell(J)=\ell(I):\ J\subset I_0}\left\|\Delta_J^\omega g\right\|_{L^2(\omega)}\sqrt{|J|_\omega} \\
&\leq 2^{-(1-\varepsilon(n+1-\alpha))s}\left|\mathbb{E}_{I_\theta}^\sigma\Delta_I^\sigma f\right|\ \mathrm{P}^\alpha(I_0,\mathbf{1}_{I_\theta}\sigma)\sqrt{|I_0|_\omega}\Lambda(I,I_0,I_\theta,s), \\
\Lambda(I,I_0,I_\theta,s)^2 &\equiv \sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}:\ 2^s\ell(J)=\ell(I):\ J\subset I_0}\left\|\Delta_J^\omega g\right\|_{L^2(\omega)}^2 .
\end{aligned}$$

The last line follows upon using the Cauchy-Schwarz inequality and the fact that $\Delta_J^\omega g=0$ if $J\notin\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}$. We also note that since $2^{s+1}\ell(J)=\ell(I)$,

$$\begin{aligned}
\sum_{I_0\in\mathfrak{C}_{\Omega\mathcal{D}}(I)}\Lambda(I,I_0,I_\theta,s)^2 &\equiv \sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}:\ 2^{s+1}\ell(J)=\ell(I):\ J\subset I}\left\|\Delta_J^\omega g\right\|_{L^2(\omega)}^2 ; \\
\sum_{I\in\mathcal{C}_A}\sum_{I_0\in\mathfrak{C}_{\Omega\mathcal{D}}(I)}\Lambda(I,I_0,I_\theta,s)^2 &\leq \left\|\mathsf{P}_{\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}}^\omega g\right\|_{L^2(\omega)}^2 .
\end{aligned}\tag{7.14}$$

Using

$$\left|\mathbb{E}_{I_\theta}^\sigma\Delta_I^\sigma f\right|\leq\sqrt{\mathbb{E}_{I_\theta}^\sigma\left|\Delta_I^\sigma f\right|^2}\leq\left\|\Delta_I^\sigma f\right\|_{L^2(\sigma)}\ |I_\theta|_\sigma^{-\frac{1}{2}},\tag{7.15}$$

we can thus estimate $A(I,I_0,I_\theta,s)$ as follows, in which we use the A_2^α hypothesis $\sup_I\frac{|I|_\sigma|I|_\omega}{|I|^{2\left(1-\frac{\alpha}{n}\right)}}=A_2^\alpha<\infty$:

$$\begin{aligned}
A(I,I_0,I_\theta,s) &\lesssim 2^{-(1-\varepsilon(n+1-\alpha))s}\left\|\Delta_I^\sigma f\right\|_{L^2(\sigma)}\Lambda(I,I_0,I_\theta,s)\cdot|I_\theta|_\sigma^{-\frac{1}{2}}\,\mathrm{P}^\alpha(I_0,\mathbf{1}_{I_\theta}\sigma)\sqrt{|I_0|_\omega} \\
&\lesssim \sqrt{A_2^\alpha}2^{-(1-\varepsilon(n+1-\alpha))s}\left\|\Delta_I^\sigma f\right\|_{L^2(\sigma)}\Lambda(I,I_0,I_\theta,s),
\end{aligned}$$

since $\mathrm{P}^\alpha(I_0,\mathbf{1}_{I_\theta}\sigma)\lesssim\frac{|I_\theta|_\sigma}{|I_\theta|^{1-\frac{\alpha}{n}}}$ shows that

$$|I_\theta|_\sigma^{-\frac{1}{2}}\,\mathrm{P}^\alpha(I_0,\mathbf{1}_{I_\theta}\sigma)\ \sqrt{|I_0|_\omega}\lesssim\frac{\sqrt{|I_\theta|_\sigma}\sqrt{|I_0|_\omega}}{|I|^{1-\frac{\alpha}{n}}}\lesssim\sqrt{A_2^\alpha}.$$

An application of Cauchy-Schwarz to the sum over I using (7.14) then shows that

$$\begin{aligned}
&\sum_{I\in\mathcal{C}_A}\sum_{\substack{I_0,I_\theta\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\\ I_0\neq I_\theta}}A(I,I_0,I_\theta,s) \\
&\lesssim \sqrt{A_2^\alpha}2^{-(1-\varepsilon(n+1-\alpha))s}\sqrt{\sum_{I\in\mathcal{C}_A}\left\|\Delta_I^\sigma f\right\|_{L^2(\sigma)}^2}\sqrt{\sum_{I\in\mathcal{C}_A}\left(\sum_{\substack{I_0,I_\theta\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\\ I_0\neq I_\theta}}\Lambda(I,I_0,I_\theta,s)\right)^2} \\
&\lesssim \sqrt{A_2^\alpha}2^{-(1-\varepsilon(n+1-\alpha))s}\left\|\mathsf{P}_{\mathcal{C}_A}^\sigma f\right\|_{L^2(\sigma)}\sqrt{2^n}\sqrt{\sum_{I\in\mathcal{C}_A}\left(\sum_{\substack{I_0\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\\ I_0\neq I_\theta}}\Lambda(I,I_0,I_\theta,s)\right)^2} \\
&\lesssim \sqrt{A_2^\alpha}2^{-(1-\varepsilon(n+1-\alpha))s}\left\|\mathsf{P}_{\mathcal{C}_A}^\sigma f\right\|_{L^2(\sigma)}\left\|\mathsf{P}_{\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}}^\omega g\right\|_{L^2(\omega)} .
\end{aligned}$$

This estimate is summable in $s\geq\mathbf{r}$, and so the proof of

$$\begin{aligned}\left|\mathsf{B}_{neighbour}^{A}\left(f,g\right)\right| &= \left|\sum_{\substack{I\in\mathcal{C}_{A}\text{ and }J\in\mathcal{C}_{A}^{\boldsymbol{\tau}-\mathrm{shift}}\\ J\Subset_{\boldsymbol{\rho},\varepsilon}I}}\sum_{\theta(I_J)\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\setminus\{I_J\}}\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{\theta(I_J)}\bigtriangleup_{I}^{\sigma}f\right),\bigtriangleup_{J}^{\omega}g\right\rangle_{\omega}\right| \\ &= \left|\sum_{I\in\mathcal{C}_{A}}\sum_{\substack{I_0,I_\theta\in\mathfrak{C}_{\Omega\mathcal{D}}(I)\\ I_0\neq I_\theta}}\sum_{s=\mathbf{r}}^{\infty}A(I,I_0,I_\theta,s)\right| \\ &\lesssim \sqrt{A_2^{\alpha}}\left\|\mathsf{P}_{\mathcal{C}_A}^{\sigma}f\right\|_{L^2(\sigma)}\left\|\mathsf{P}_{\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}}^{\omega}g\right\|_{L^2(\omega)}\end{aligned}$$

is complete.

It is to the sublinear form on the left side of (10.7) below, derived from the stopping form $\mathsf{B}_{stop}^{A}\left(f,g\right)$, that the argument of M. Lacey in [Lac] will be adapted. This will result in the inequality

$$\left|\mathsf{B}_{stop}^{A}\left(f,g\right)\right|\lesssim\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}+\sqrt{A_2^{\alpha}}+\sqrt{A_2^{\alpha,\mathrm{punct}}}\right)\left(\alpha_{\mathcal{A}}\left(A\right)\sqrt{\left|A\right|_{\sigma}}+\left\|f\right\|_{L^2(\sigma)}\right)\left\|g\right\|_{L^2(\omega)}\ ,\qquad A\in\mathcal{A}, \tag{7.16}$$

where the bounded quasiaverages of f in $\mathsf{B}_{stop}^{A}\left(f,g\right)$ will prove crucial. But first we turn to completing the proof of the bound (7.10) for the far below form $\mathsf{T}_{\text{far below}}\left(f,g\right)$ using the Intertwining Proposition and the control of functional energy by the $\mathcal{A}_2^{\alpha}$ condition and the energy condition $\mathcal{E}_{\alpha}$.

8. Intertwining proposition

Here we generalize the Intertwining Proposition (see e.g. [Expanded] and [SaShUr5]) to higher dimensions. The main principle here says that, modulo terms that are controlled by the γ-functional quasienergy constant $\mathfrak{F}_{\alpha}$ and the quasiNTV constant $\mathcal{NTV}_{\alpha}$ (see below), we can pass the shifted ω-corona projection $\mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}^{\omega}$ through the operator T^{α} to become the shifted σ-corona projection $\mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}^{\sigma}$, provided the goodness parameters $\mathbf{r},\varepsilon$ are chosen sufficiently large and small respectively depending on γ. More precisely, the idea is that with $T_{\sigma}^{\alpha}f\equiv T^{\alpha}\left(f\sigma\right)$, the intertwining operator

$$\mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}^{\omega}\left[\mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}^{\omega}T_{\sigma}^{\alpha}-T_{\sigma}^{\alpha}\mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}^{\sigma}\right]\mathsf{P}_{\mathcal{C}_A}^{\sigma}$$

is bounded with constant $\mathfrak{F}_{\alpha}+\mathcal{NTV}_{\alpha}$, provided $\gamma\leq c_n 2^{(1-\varepsilon)\mathbf{r}}$. In those cases where the coronas $\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}$ and $\mathcal{C}_A$ are (almost) disjoint, the intertwining operator reduces (essentially) to $\mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}-\mathrm{shift}}}^{\omega}T_{\sigma}^{\alpha}\mathsf{P}_{\mathcal{C}_A}^{\sigma}$, and then combined with the control of the functional quasienergy constant $\mathfrak{F}_{\alpha}$ by the quasienergy condition constant $\mathcal{E}_{\alpha}$ and $\mathcal{A}_2^{\alpha}+\mathcal{A}_2^{\alpha,*}+A_2^{\alpha,\mathrm{punct}}$, provided γ is sufficiently large depending only on n and α, we obtain the required bound (7.10) for $\mathsf{T}_{\text{far below}}\left(f,g\right)$ above.

To describe the quantities we use to bound these forms, we need to adapt to higher dimensions three definitions used for the Hilbert transform that are relevant to functional energy.

Definition 14. *A collection $\mathcal{F}$ of dyadic quasicubes is σ-*Carleson *if*

$$\sum_{F\in\mathcal{F}:\ F\subset S}\left|F\right|_{\sigma}\leq C_{\mathcal{F}}\left|S\right|_{\sigma},\qquad S\in\mathcal{F}.$$

The constant $C_{\mathcal{F}}$ is referred to as the Carleson norm of $\mathcal{F}$.

Definition 15. *Let $\mathcal{F}$ be a collection of dyadic quasicubes. The good τ-shifted corona corresponding to F is defined by*

$$\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}\equiv\left\{J\in\Omega\mathcal{D}_{\mathrm{good}}^{\omega}:J\Subset_{\boldsymbol{\tau},\varepsilon}F\text{ and }J\not\Subset_{\boldsymbol{\tau},\varepsilon}F'\text{ for any }F'\in\mathfrak{C}_{\mathcal{F}}\left(F\right)\right\}.$$

Note that $\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}=\mathcal{C}_F^{\boldsymbol{\tau}-\mathrm{shift}}\cap\Omega\mathcal{D}_{\mathrm{good}}^{\omega}$, and that the collections $\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}$ have bounded overlap $\boldsymbol{\tau}$ since for fixed J, there are at most $\boldsymbol{\tau}$ quasicubes $F\in\mathcal{F}$ with the property that J is good and $J\Subset_{\boldsymbol{\tau},\varepsilon}F$ and $J\not\Subset_{\boldsymbol{\tau},\varepsilon}F'$ for any $F'\in\mathfrak{C}_{\mathcal{F}}\left(F\right)$. Here $\mathfrak{C}_{\mathcal{F}}\left(F\right)$ denotes the set of $\mathcal{F}$-children of F. Given any collection

$\mathcal{H} \subset \Omega\mathcal{D}$ of quasicubes, and a dyadic quasicube J, we define the corresponding quasiHaar projection $\mathsf{P}_{\mathcal{H}}^{\omega}$ and its localization $\mathsf{P}_{\mathcal{H};J}^{\omega}$ to J by

$$\mathsf{P}_{\mathcal{H}}^{\omega} = \sum_{H \in \mathcal{H}} \triangle_H^{\omega} \text{ and } \mathsf{P}_{\mathcal{H};J}^{\omega} = \sum_{H \in \mathcal{H}:\ H \subset J} \triangle_H^{\omega} \ . \tag{8.1}$$

Definition 16. *Let $\mathfrak{F}_\alpha$ be the smallest constant in the '**f**unctional quasienergy' inequality below, holding for all $h \in L^2(\sigma)$ and all σ-Carleson collections $\mathcal{F}$ with Carleson norm $C_{\mathcal{F}}$ bounded by a fixed constant C:*

$$\sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)} \left(\frac{\mathrm{P}^\alpha(J, h\sigma)}{|J|^{\frac{1}{n}}} \right)^2 \left\| \mathsf{P}^{\omega}_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}};J} \mathbf{x} \right\|^2_{L^2(\omega)} \leq \mathfrak{F}_\alpha \|h\|_{L^2(\sigma)} \ . \tag{8.2}$$

This definition of $\mathfrak{F}_\alpha$ depends also on the choice of the fixed constant C, but it will be clear from the arguments below that C may be taken to depend only on n and α, and we do not compute its value here. There is a similar definition of the dual constant $\mathfrak{F}_\alpha^*$.

Remark 10. *If in (8.2), we take $h = \mathbf{1}_I$ and $\mathcal{F}$ to be the trivial Carleson collection $\{I_r\}_{r=1}^{\infty}$ where the quasicubes I_r are pairwise disjoint in I, then we obtain the deep quasienergy condition in Definition 8, but with $\mathsf{P}^{\omega}_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}};J}$ in place of $\mathsf{P}_J^{\mathrm{subgood},\omega}$. However, the projection $\mathsf{P}_J^{\mathrm{subgood},\omega}$ is larger than $\mathsf{P}^{\omega}_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}};J}$ by the finite projection $\sum_{2^{-\boldsymbol{\tau}}\ell(J) \leq \ell(J') \leq 2^{-\mathbf{r}}\ell(J)} \triangle_{J'}^{\omega}$, and so we just miss obtaining the deep quasienergy condition as a consequence of the functional quasienergy condition. Nevertheless, this near miss with $h = \mathbf{1}_I$ explains the terminology 'functional' quasienergy.*

We now show that the functional quasienergy inequality (8.2) suffices to prove an α-fractional n-dimensional analogue of the Intertwining Proposition (see e.g. [Expanded]). Let $\mathcal{F}$ be any subset of $\Omega\mathcal{D}$. For any $J \in \Omega\mathcal{D}$, we define $\pi^0_{\mathcal{F}} J$ to be the smallest $F \in \mathcal{F}$ that contains J. Then for $s \geq 1$, we recursively define $\pi^s_{\mathcal{F}} J$ to be the smallest $F \in \mathcal{F}$ that *strictly* contains $\pi^{s-1}_{\mathcal{F}} J$. This definition satisfies $\pi^{s+t}_{\mathcal{F}} J = \pi^s_{\mathcal{F}} \pi^t_{\mathcal{F}} J$ for all $s, t \geq 0$ and $J \in \Omega\mathcal{D}$. In particular $\pi^s_{\mathcal{F}} J = \pi^s_{\mathcal{F}} F$ where $F = \pi^0_{\mathcal{F}} J$. In the special case $\mathcal{F} = \Omega\mathcal{D}$ we often suppress the subscript $\mathcal{F}$ and simply write π^s for $\pi^s_{\Omega\mathcal{D}}$. Finally, for $F \in \mathcal{F}$, we write $\mathfrak{C}_{\mathcal{F}}(F) \equiv \left\{ F' \in \mathcal{F} : \pi^1_{\mathcal{F}} F' = F \right\}$ for the collection of $\mathcal{F}$-children of F. Let

$$\mathcal{NTV}_\alpha \equiv \sqrt{\mathcal{A}_2^\alpha} + \mathfrak{T}_\alpha + \mathcal{WBP}_\alpha,$$

where we remind the reader that $\mathfrak{T}_\alpha$ and $\mathcal{WBP}_\alpha$ refer to the quasitesting condition and quasiweak boundedness property respectively.

Proposition 1 (The Intertwining Proposition)**.** *Suppose that $\mathcal{F}$ is σ-Carleson. Then*

$$\left| \sum_{F \in \mathcal{F}} \sum_{I:\ I \supsetneqq F} \left\langle T^\alpha_\sigma \triangle^\sigma_I f, \mathsf{P}^{\omega}_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}} g \right\rangle_\omega \right| \lesssim (\mathfrak{F}_\alpha + \mathcal{E}_\alpha + \mathcal{NTV}_\alpha) \ \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} \ .$$

Proof. We write the left hand side of the display above as

$$\sum_{F \in \mathcal{F}} \sum_{I:\ I \supsetneqq F} \langle T^\alpha_\sigma \triangle^\sigma_I f, g_F \rangle_\omega = \sum_{F \in \mathcal{F}} \left\langle T^\alpha_\sigma \left(\sum_{I:\ I \supsetneqq F} \triangle^\sigma_I f \right), g_F \right\rangle_\omega \equiv \sum_{F \in \mathcal{F}} \langle T^\alpha_\sigma f_F, g_F \rangle_\omega \ ,$$

where

$$g_F = \mathsf{P}^{\omega}_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}} g \text{ and } f_F \equiv \sum_{I:\ I \supsetneqq F} \triangle^\sigma_I f \ .$$

Note that g_F is supported in F, and that f_F is constant on F. We note that the quasicubes I occurring in this sum are linearly and consecutively ordered by inclusion, along with the quasicubes $F' \in \mathcal{F}$ that contain F. More precisely, we can write

$$F \equiv F_0 \subsetneqq F_1 \subsetneqq F_2 \subsetneqq ... \subsetneqq F_n \subsetneqq F_{n+1} \subsetneqq ...F_N$$

where $F_m = \pi^m_{\mathcal{F}} F$ for all $m \geq 1$. We can also write

$$F = F_0 \subsetneqq I_1 \subsetneqq I_2 \subsetneqq ... \subsetneqq I_k \subsetneqq I_{k+1} \subsetneqq ... \subsetneqq I_K = F_N$$

where $I_k = \pi^k_{\Omega\mathcal{D}} F$ for all $k \geq 1$. There is a (unique) subsequence $\{k_m\}_{m=1}^N$ such that

$$F_m = I_{k_m}, \qquad 1 \leq m \leq N.$$

Define

$$f_F(x) = \sum_{\ell=1}^{\infty} \triangle^\sigma_{I_\ell} f(x).$$

Assume now that $k_m \leq k < k_{m+1}$. We denote the $2^n - 1$ siblings of I by $\theta(I)$, $\theta \in \Theta$, i.e. $\{\theta(I)\}_{\theta\in\Theta} = \mathfrak{C}_{\Omega\mathcal{D}}(\pi_{\Omega\mathcal{D}} I) \setminus \{I\}$. There are two cases to consider here:

$$\theta(I_k) \notin \mathcal{F} \text{ and } \theta(I_k) \in \mathcal{F}.$$

Suppose first that $\theta(I_k) \notin \mathcal{F}$. Then $\theta(I_k) \in \mathcal{C}^\sigma_{F_{m+1}}$ and using a telescoping sum, we compute that for

$$x \in \theta(I_k) \subset I_{k+1} \setminus I_k \subset F_{m+1} \setminus F_m,$$

we have

$$|f_F(x)| = \left|\sum_{\ell=k}^{\infty} \triangle^\sigma_{I_\ell} f(x)\right| = \left|\mathbb{E}^\sigma_{\theta(I_k)} f - \mathbb{E}^\sigma_{I_K} f\right| \lesssim \mathbb{E}^\sigma_{F_{m+1}} |f| \ .$$

On the other hand, if $\theta(I_k) \in \mathcal{F}$, then $I_{k+1} \in \mathcal{C}^\sigma_{F_{m+1}}$ and we have

$$\left|f_F(x) - \triangle^\sigma_{\theta(I_k)} f(x)\right| = \left|\sum_{\ell=k+1}^{\infty} \triangle^\sigma_{I_\ell} f(x)\right| = \left|\mathbb{E}^\sigma_{I_{k+1}} f - \mathbb{E}^\sigma_{I_K} f\right| \lesssim \mathbb{E}^\sigma_{F_{m+1}} |f| \ .$$

Now we write

$$\begin{aligned}
f_F &= \varphi_F + \psi_F, \\
\varphi_F &\equiv \sum_{k,\theta:\ \theta(I_k)\in\mathcal{F}}^{\infty} \mathbf{1}_{\theta(I_k)} \triangle^\sigma_{I_k} f \text{ and } \psi_F = f_F - \varphi_F \ ; \\
\sum_{F\in\mathcal{F}} \langle T^\alpha_\sigma f_F, g_F\rangle_\omega &= \sum_{F\in\mathcal{F}} \langle T^\alpha_\sigma \varphi_F, g_F\rangle_\omega + \sum_{F\in\mathcal{F}} \langle T^\alpha_\sigma \psi_F, g_F\rangle_\omega \ ,
\end{aligned}$$

and note that both φ_F and ψ_F are constant on F. We can apply (6.4) using $\theta(I_k) \in \mathcal{F}$ to the first sum here to obtain

$$\begin{aligned}
\left|\sum_{F\in\mathcal{F}} \langle T^\alpha_\sigma \varphi_F, g_F\rangle_\omega\right| &\lesssim \mathcal{NTV}_\alpha \left\|\sum_{F\in\mathcal{F}} \varphi_F\right\|_{L^2(\sigma)} \left\|\sum_{F\in\mathcal{F}} g_F\right\|^2_{L^2(\omega)} \\
&\lesssim \mathcal{NTV}_\alpha \|f\|_{L^2(\sigma)} \left[\sum_{F\in\mathcal{F}} \|g_F\|^2_{L^2(\omega)}\right]^{\frac{1}{2}}.
\end{aligned}$$

Turning to the second sum we note that

$$\begin{aligned}
|\psi_F| &\leq \sum_{m=0}^{N} \left(\mathbb{E}^\sigma_{F_{m+1}} |f|\right) \mathbf{1}_{F_{m+1}\setminus F_m} = \left(\mathbb{E}^\sigma_F |f|\right) \mathbf{1}_F + \sum_{m=0}^{N} \left(\mathbb{E}^\sigma_{\pi^{m+1}_{\mathcal{F}} F} |f|\right) \mathbf{1}_{\pi^{m+1}_{\mathcal{F}} F \setminus \pi^m_{\mathcal{F}} F} \\
&= \left(\mathbb{E}^\sigma_F |f|\right) \mathbf{1}_F + \sum_{F'\in\mathcal{F}:\ F\subset F'} \left(\mathbb{E}^\sigma_{\pi_{\mathcal{F}} F'} |f|\right) \mathbf{1}_{\pi_{\mathcal{F}} F' \setminus F'} \\
&\leq \alpha_{\mathcal{F}}(F) \mathbf{1}_F + \sum_{F'\in\mathcal{F}:\ F\subset F'} \alpha_{\mathcal{F}}(\pi_{\mathcal{F}} F') \mathbf{1}_{\pi_{\mathcal{F}} F' \setminus F'} \\
&\leq \alpha_{\mathcal{F}}(F) \mathbf{1}_F + \sum_{F'\in\mathcal{F}:\ F\subset F'} \alpha_{\mathcal{F}}(\pi_{\mathcal{F}} F') \mathbf{1}_{\pi_{\mathcal{F}} F'} \mathbf{1}_{F^c} \\
&= \alpha_{\mathcal{F}}(F) \mathbf{1}_F + \Phi \mathbf{1}_{F^c} \ , \qquad \text{for all } F \in \mathcal{F},
\end{aligned}$$

where

$$\Phi \equiv \sum_{F''\in\mathcal{F}} \alpha_{\mathcal{F}}(F'') \mathbf{1}_{F''} \ .$$

Now we write

$$\sum_{F\in\mathcal{F}}\left\langle T_{\sigma}^{\alpha}\psi_{F},g_{F}\right\rangle_{\omega}=\sum_{F\in\mathcal{F}}\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{F}\psi_{F}\right),g_{F}\right\rangle_{\omega}+\sum_{F\in\mathcal{F}}\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{F^{c}}\psi_{F}\right),g_{F}\right\rangle_{\omega}\equiv I+II.$$

Then quasicube testing $\left|\left\langle T_{\sigma}^{\alpha}\mathbf{1}_{F},g_{F}\right\rangle_{\omega}\right|=\left|\left\langle \mathbf{1}_{F}T_{\sigma}^{\alpha}\mathbf{1}_{F},g_{F}\right\rangle_{\omega}\right|\leq\mathfrak{T}_{T^{\alpha}}\sqrt{\left|F\right|_{\sigma}}\left\|g_{F}\right\|_{L^{2}(\omega)}$ and 'quasi' orthogonality, together with the fact that ψ_{F} is a constant on F bounded by $\alpha_{\mathcal{F}}\left(F\right)$, give

$$\begin{aligned}
\left|I\right| &\leq \sum_{F\in\mathcal{F}}\left|\left\langle T_{\sigma}^{\alpha}\mathbf{1}_{F}\psi_{F},g_{F}\right\rangle_{\omega}\right|\lesssim\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}\left(F\right)\ \left|\left\langle T_{\sigma}^{\alpha}\mathbf{1}_{F},g_{F}\right\rangle_{\omega}\right| \\
&\lesssim \sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}\left(F\right)\mathcal{NTV}_{\alpha}\sqrt{\left|F\right|_{\sigma}}\left\|g_{F}\right\|_{L^{2}(\omega)}\lesssim\mathcal{NTV}_{\alpha}\left\|f\right\|_{L^{2}(\sigma)}\left[\sum_{F\in\mathcal{F}}\left\|g_{F}\right\|_{L^{2}(\omega)}^{2}\right]^{\frac{1}{2}}.
\end{aligned}$$

Now $\mathbf{1}_{F^{c}}\psi_{F}$ is supported outside F, and each J in the quasiHaar support of g_{F} is $\mathbf{r}$-deeply embedded in F, i.e. $J\Subset_{\mathbf{r},\varepsilon}F$. Thus we can apply the Energy Lemma 8 to obtain

$$\begin{aligned}
\left|II\right| &= \left|\sum_{F\in\mathcal{F}}\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{F^{c}}\psi_{F}\right),g_{F}\right\rangle_{\omega}\right| \\
&\lesssim \sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{F^{c}}\Phi\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\left\|\mathsf{P}_{\mathcal{C}_{F}^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}};J}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}\left\|\mathsf{P}_{J}^{\omega}g_{F}\right\|_{L^{2}(\omega)} \\
&\quad +\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}\frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J,\mathbf{1}_{F^{c}}\Phi\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\left\|\mathsf{P}_{\left(\mathcal{C}_{F}^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}\right)^{*};J}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}\left\|\mathsf{P}_{J}^{\omega}g_{F}\right\|_{L^{2}(\omega)} \\
&\equiv II_{G}+II_{B}\ .
\end{aligned}$$

Then from Cauchy-Schwarz, the functional quasienergy condition, and $\left\|\Phi\right\|_{L^{2}(\sigma)}\lesssim\left\|f\right\|_{L^{2}(\sigma)}$ we obtain

$$\begin{aligned}
\left|II_{G}\right| &\leq \left(\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{F^{c}}\Phi\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\left\|\mathsf{P}_{\mathcal{C}_{F}^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}};J}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}^{2}\right)^{\frac{1}{2}} \\
&\quad\times\left(\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}\left\|\mathsf{P}_{J}^{\omega}g_{F}\right\|_{L^{2}(\omega)}^{2}\right)^{\frac{1}{2}} \\
&\lesssim \mathfrak{F}_{\alpha}\left\|\Phi\right\|_{L^{2}(\sigma)}\left[\sum_{F\in\mathcal{F}}\left\|g_{F}\right\|_{L^{2}(\omega)}^{2}\right]^{\frac{1}{2}}\lesssim\boldsymbol{\tau}\mathfrak{F}_{\alpha}\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)},
\end{aligned}$$

by the $\boldsymbol{\tau}$-overlap (7.8) of the shifted coronas $\mathcal{C}_{F}^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}$.

In term II_{B} the projections $\mathsf{P}_{\left(\mathcal{C}_{F}^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}\right)^{*};J}^{\omega}$ are no longer almost orthogonal, and we must instead exploit the decay in the Poisson integral $\mathrm{P}_{1+\delta'}^{\alpha}$ along with goodness of the quasicubes J. This idea was already used by M. Lacey and B. Wick in [LaWi] in a similar situation. As a consequence of this decay we will be able to bound II_{B} *directly* by the quasienergy condition, without having to invoke the more difficult functional quasienergy condition. For the decay we compute

$$\begin{aligned}
\frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J,\Phi\sigma\right)}{\left|J\right|^{\frac{1}{n}}} &= \int_{F^{c}}\frac{\left|J\right|^{\frac{\delta'}{n}}}{\left|y-c_{J}\right|^{n+1+\delta-\alpha}}\Phi\left(y\right)d\sigma\left(y\right) \\
&\leq \sum_{t=0}^{\infty}\int_{\pi_{\mathcal{F}}^{t+1}F\setminus\pi_{\mathcal{F}}^{t}F}\left(\frac{\left|J\right|^{\frac{1}{n}}}{\mathrm{dist}\left(c_{J},\left(\pi_{\mathcal{F}}^{t}F\right)^{c}\right)}\right)^{\delta'}\frac{1}{\left|y-c_{J}\right|^{n+1-\alpha}}\Phi\left(y\right)d\sigma\left(y\right) \\
&\leq \sum_{t=0}^{\infty}\left(\frac{\left|J\right|^{\frac{1}{n}}}{\mathrm{dist}\left(c_{J},\left(\pi_{\mathcal{F}}^{t}F\right)^{c}\right)}\right)^{\delta'}\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\pi_{\mathcal{F}}^{t+1}F\setminus\pi_{\mathcal{F}}^{t}F}\Phi\sigma\right)}{\left|J\right|^{\frac{1}{n}}},
\end{aligned}$$

and then use the goodness inequality

$$\operatorname{dist}\left(c_J,\left(\pi_{\mathcal{F}}^t F\right)^c\right)\geq\frac{1}{2}\ell\left(\pi_{\mathcal{F}}^t F\right)^{1-\varepsilon}\ell\left(J\right)^{\varepsilon}\geq\frac{1}{2}2^{t(1-\varepsilon)}\ell\left(F\right)^{1-\varepsilon}\ell\left(J\right)^{\varepsilon}\geq2^{t(1-\varepsilon)-1}\ell\left(J\right),$$

to conclude that

$$\begin{aligned}\left(\frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J,\mathbf{1}_{F^c}\Phi\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2 &\lesssim \left(\sum_{t=0}^{\infty}2^{-t\delta'(1-\varepsilon)}\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\pi_{\mathcal{F}}^{t+1}F\setminus\pi_{\mathcal{F}}^{t}F}\Phi\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\\ &\lesssim \sum_{t=0}^{\infty}2^{-t\delta'(1-\varepsilon)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\pi_{\mathcal{F}}^{t+1}F\setminus\pi_{\mathcal{F}}^{t}F}\Phi\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2 .\end{aligned}\tag{8.3}$$

Now we apply Cauchy-Schwarz to obtain

$$\begin{aligned}II_B &= \sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}\frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J,\mathbf{1}_{F^c}\Phi\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\left\|\mathsf{P}_{\left(\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}\right)^*;J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}\left\|\mathsf{P}_J^{\omega}g_F\right\|_{L^2(\omega)}\\ &\leq\left(\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}\left(\frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J,\mathbf{1}_{F^c}\Phi\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_{\left(\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}\right)^*;J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\right)^{\frac{1}{2}}\left[\sum_F\left\|g_F\right\|_{L^2(\omega)}^2\right]^{\frac{1}{2}}\\ &\equiv\sqrt{II_{\mathrm{energy}}}\left[\sum_F\left\|g_F\right\|_{L^2(\omega)}^2\right]^{\frac{1}{2}},\end{aligned}$$

and it remains to estimate II_{energy}. From (8.3) and the plugged deep quasienergy condition we have

$$\begin{aligned}&II_{\mathrm{energy}}\\ &\leq\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}\sum_{t=0}^{\infty}2^{-t\delta'(1-\varepsilon)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\pi_{\mathcal{F}}^{t+1}F\setminus\pi_{\mathcal{F}}^{t}F}\Phi\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_{\left(\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}\right)^*;J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\\ &=\sum_{t=0}^{\infty}2^{-t\delta'(1-\varepsilon)}\sum_{G\in\mathcal{F}}\sum_{F\in\mathfrak{C}_{\mathcal{F}}^{(t+1)}(G)}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{G\setminus\pi_{\mathcal{F}}^{t}F}\Phi\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_{\left(\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}\right)^*;J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\\ &\lesssim\sum_{t=0}^{\infty}2^{-t\delta'(1-\varepsilon)}\sum_{G\in\mathcal{F}}\alpha_{\mathcal{F}}\left(G\right)^2\sum_{F\in\mathfrak{C}_{\mathcal{F}}^{(t+1)}(G)}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{G\setminus\pi_{\mathcal{F}}^{t}F}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\\ &\lesssim\sum_{t=0}^{\infty}2^{-t\delta'(1-\varepsilon)}\sum_{G\in\mathcal{F}}\alpha_{\mathcal{F}}\left(G\right)^2\left(\mathcal{E}_{\alpha}^2+A_2^{\alpha}\right)\left|G\right|_{\sigma}\lesssim\left(\mathcal{E}_{\alpha}^2+A_2^{\alpha}\right)\left\|f\right\|_{L^2(\sigma)}^2 .\end{aligned}$$

This completes the proof of the Intertwining Proposition 1. □

9. Control of functional energy by energy modulo $\mathcal{A}_2^{\alpha}$ and $A_2^{\alpha,\mathrm{punct}}$

Now we arrive at one of our main propositions in the proof of our theorem. We show that the functional quasienergy constants $\mathfrak{F}_{\alpha}$ as in (8.2) are controlled by $\mathcal{A}_2^{\alpha}$, $A_2^{\alpha,\mathrm{punct}}$ and both the *deep* and *refined* quasienergy constants $\mathcal{E}_{\alpha}^{\mathrm{deep}}$ and $\mathcal{E}_{\alpha}^{\mathrm{refined}}$ defined in Definition 8. Recall $\left(\mathcal{E}_{\alpha}\right)^2=\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}\right)^2+\left(\mathcal{E}_{\alpha}^{\mathrm{refined}}\right)^2$. The proof of this fact is further complicated when common point masses are permitted, accounting for the inclusion of the punctured Muckenhoupt condition $A_2^{\alpha,\mathrm{punct}}$.

Proposition 2. *We have*

$$\mathfrak{F}_{\alpha}\lesssim\mathcal{E}_{\alpha}^{\mathrm{plug}}+\sqrt{\mathcal{A}_2^{\alpha}}+\sqrt{\mathcal{A}_2^{\alpha,*}}+\sqrt{A_2^{\alpha,\mathrm{punct}}}\ \text{ and }\ \mathfrak{F}_{\alpha}^*\lesssim\mathcal{E}_{\alpha}^{\mathrm{plug},*}+\sqrt{\mathcal{A}_2^{\alpha}}+\sqrt{\mathcal{A}_2^{\alpha,*}}+\sqrt{A_2^{\alpha,*,\mathrm{punct}}} .$$

To prove this proposition, we fix $\mathcal{F}$ as in (8.2), and set

$$\mu \equiv \sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)} \left\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)} \cdot \delta_{(c_J,\ell(J))} \text{ and } d\overline{\mu}(x,t) \equiv \frac{1}{t^2} d\mu(x,t) \ , \tag{9.1}$$

where $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)$ consists of the maximal $\mathbf{r}$-deeply embedded subquasicubes of F, and where $\delta_{(c_J,\ell(J))}$ denotes the Dirac unit mass at the point $(c_J, \ell(J))$ in the upper half-space $\mathbb{R}^{n+1}_+$. Here J is a dyadic quasicube with center c_J and side length $\ell(J)$. For convenience in notation, we denote for any dyadic quasicube J the localized projection $\mathsf{P}^{\omega}_{\mathcal{C}^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}_F;J}$ given in (8.1) by

$$\mathsf{P}^{\omega}_{F,J} \equiv \mathsf{P}^{\omega}_{\mathcal{C}^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}_F;J} = \sum_{J'\subset J:\ J'\in\mathcal{C}^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}_F} \triangle^{\omega}_{J'}.$$

We emphasize that the quasicubes $J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)$ are not necessarily good, but that the subquasicubes $J' \subset J$ arising in the projection $\mathsf{P}^{\omega}_{F,J}$ are good. We can replace $\mathbf{x}$ by $\mathbf{x}-\mathbf{c}$ inside the projection for any choice of $\mathbf{c}$ we wish; the projection is unchanged. More generally, δ_q denotes a Dirac unit mass at a point q in the upper half-space $\mathbb{R}^{n+1}_+$.

We prove the two-weight inequality

$$\left\|\mathbb{P}^{\alpha}(f\sigma)\right\|_{L^2(\mathbb{R}^{n+1}_+,\overline{\mu})} \lesssim \left(\mathcal{E}^{\mathrm{plug}}_{\alpha} + \sqrt{\mathcal{A}^{\alpha}_2} + \sqrt{\mathcal{A}^{\alpha,*}_2} + \sqrt{A^{\alpha,\mathrm{punct}}_2}\right) \|f\|_{L^2(\sigma)}\,, \tag{9.2}$$

for all nonnegative f in $L^2(\sigma)$, noting that $\mathcal{F}$ and f are *not* related here. Above, $\mathbb{P}^{\alpha}(\cdot)$ denotes the α-fractional Poisson extension to the upper half-space $\mathbb{R}^{n+1}_+$,

$$\mathbb{P}^{\alpha}\nu(x,t) \equiv \int_{\mathbb{R}^n} \frac{t}{\left(t^2+|x-y|^2\right)^{\frac{n+1-\alpha}{2}}} d\nu(y)\,,$$

so that in particular

$$\left\|\mathbb{P}^{\alpha}(f\sigma)\right\|^2_{L^2(\mathbb{R}^{n+1}_+,\overline{\mu})} = \sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(F)} \mathbb{P}^{\alpha}(f\sigma)\left(c(J),\ell(J)\right)^2 \left\|\mathsf{P}^{\omega}_{F,J}\frac{x}{|J|^{\frac{1}{n}}}\right\|^2_{L^2(\omega)}\,,$$

and so (9.2) proves the first line in Proposition 2 upon inspecting (8.2). Note also that we can equivalently write $\left\|\mathbb{P}^{\alpha}(f\sigma)\right\|_{L^2(\mathbb{R}^{n+1}_+,\overline{\mu})} = \left\|\widetilde{\mathbb{P}}^{\alpha}(f\sigma)\right\|_{L^2(\mathbb{R}^{n+1}_+,\mu)}$ where $\widetilde{\mathbb{P}}^{\alpha}\nu(x,t) \equiv \frac{1}{t}\mathbb{P}^{\alpha}\nu(x,t)$ is the renormalized Poisson operator. Here we have simply shifted the factor $\frac{1}{t^2}$ in $\overline{\mu}$ to $\left|\widetilde{\mathbb{P}}^{\alpha}(f\sigma)\right|^2$ instead, and we will do this shifting often throughout the proof when it is convenient to do so.

The characterization of the two-weight inequality for fractional and Poisson integrals in [Saw] was stated in terms of the collection $\mathcal{P}^n$ of cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. It is a routine matter to pullback the Poisson inequality under a globally biLipschitz map $\Omega : \mathbb{R}^n \to \mathbb{R}^n$, then apply the theorem in [Saw] (as a black box), and then to pushforward the conclusions of the theorems so as to extend these characterizations of fractional and Poisson integral inequalities to the setting of quasicubes $Q \in \Omega\mathcal{P}^n$ and quasitents $Q \times [0, \ell(Q)] \subset \mathbb{R}^{n+1}_+$ with $Q \in \Omega\mathcal{P}^n$. Using this extended theorem for the two-weight Poisson inequality, we see that inequality (9.2) requires checking these two inequalities for dyadic quasicubes $I \in \Omega\mathcal{D}$ and quasiboxes $\widehat{I} = I \times [0, \ell(I))$ in the upper half-space$\mathbb{R}^{n+1}_+$:

$$\int_{\mathbb{R}^{n+1}_+} \mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)(x,t)^2\, d\overline{\mu}(x,t) \equiv \left\|\mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)\right\|^2_{L^2(\widehat{I},\overline{\mu})} \lesssim \left(\left(\mathcal{E}^{\mathrm{plug}}_{\alpha}\right)^2 + \mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,*}_2 + A^{\alpha,\mathrm{punct}}_2\right)\sigma(I)\,, \tag{9.3}$$

$$\int_{\mathbb{R}^n} [\mathbb{Q}^{\alpha}(t\mathbf{1}_{\widehat{I}}\overline{\mu})]^2 d\sigma(x) \lesssim \left(\left(\mathcal{E}^{\mathrm{plug}}_{\alpha}\right)^2 + \mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,*}_2 + A^{\alpha,\mathrm{punct}}_2\right) \int_{\widehat{I}} t^2 d\overline{\mu}(x,t), \tag{9.4}$$

for all *dyadic* quasicubes $I \in \Omega\mathcal{D}$, and where the dual Poisson operator $\mathbb{Q}^{\alpha}$ is given by

$$\mathbb{Q}^{\alpha}(t\mathbf{1}_{\widehat{I}}\overline{\mu})(x) = \int_{\widehat{I}} \frac{t^2}{(t^2+|x-y|^2)^{\frac{n+1-\alpha}{2}}} d\overline{\mu}(y,t)\ .$$

It is important to note that we can choose for $\Omega\mathcal{D}$ any fixed dyadic quasigrid, the compensating point being that the integrations on the left sides of (9.3) and (9.4) are taken over the entire spaces $\mathbb{R}_+^{n+1}$ and $\mathbb{R}^n$ respectively.

Remark 11. *There is a gap in the proof of the Poisson inequality at the top of page 542 in* [Saw]. *However, this gap can be fixed as in* [SaWh] *or* [LaSaUr1].

9.1. **Poisson testing.** We now turn to proving the Poisson testing conditions (9.3) and (9.4). The same testing conditions have been considered in [SaShUr5] but in the setting of no common point masses, and the proofs there carry over to the situation here, but careful attention must now be paid to the possibility of common point masses. In [Hyt2] Hytönen circumvented this difficulty by introducing a Poisson operator 'with holes', which was then analyzed using shifted dyadic grids, but part of his argument was heavily dependent on the dimension being $n=1$, and the extension of this argument to higher dimensions is feasible (see earlier versions of this paper on the *arXiv*), but technically very involved. We circumvent the difficulty of permitting common point masses here instead by using the energy Muckenhoupt constants $A_2^{\alpha,\text{energy}}$ and $A_2^{\alpha,*,\text{energy}}$, which require control by the punctured Muckenhoupt constants $A_2^{\alpha,\text{punct}}$ and $A_2^{\alpha,*,\text{punct}}$. The following elementary Poisson inequalities (see e.g. [Vol]) will be used extensively.

Lemma 14. *Suppose that J,K,I are quasicubes in $\mathbb{R}^n$, and that μ is a positive measure supported in $\mathbb{R}^n\setminus I$. If $J\subset K\subset 2K\subset I$, then*

$$\frac{\mathrm{P}^\alpha\left(J,\mu\right)}{\left|J\right|^{\frac{1}{n}}}\lesssim\frac{\mathrm{P}^\alpha\left(K,\mu\right)}{\left|K\right|^{\frac{1}{n}}}\lesssim\frac{\mathrm{P}^\alpha\left(J,\mu\right)}{\left|J\right|^{\frac{1}{n}}},$$

while if $2J\subset K\subset I$, then

$$\frac{\mathrm{P}^\alpha\left(K,\mu\right)}{\left|K\right|^{\frac{1}{n}}}\lesssim\frac{\mathrm{P}^\alpha\left(J,\mu\right)}{\left|J\right|^{\frac{1}{n}}}.$$

Proof. We have

$$\frac{\mathrm{P}^\alpha\left(J,\mu\right)}{\left|J\right|^{\frac{1}{n}}}=\frac{1}{\left|J\right|^{\frac{1}{n}}}\int\frac{\left|J\right|^{\frac{1}{n}}}{\left(\left|J\right|^{\frac{1}{n}}+\left|x-c_J\right|\right)^{n+1-\alpha}}d\mu\left(x\right),$$

where $J\subset K\subset 2K\subset I$ implies that

$$\left|J\right|^{\frac{1}{n}}+\left|x-c_J\right|\approx\left|K\right|^{\frac{1}{n}}+\left|x-c_K\right|,\qquad x\in\mathbb{R}^n\setminus I,$$

and where $2J\subset K\subset I$ implies that

$$\left|J\right|^{\frac{1}{n}}+\left|x-c_J\right|\lesssim\left|K\right|^{\frac{1}{n}}+\left|x-c_K\right|,\qquad x\in\mathbb{R}^n\setminus I.$$

□

Now we record the bounded overlap of the projections $\mathsf{P}_{F,J}^\omega$.

Lemma 15. *Suppose $\mathsf{P}_{F,J}^\omega$ is as above and fix any $I_0\in\Omega\mathcal{D}$, so that I_0, F and J all lie in a common quasigrid. If $J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}\left(F\right)$ for some $F\in\mathcal{F}$ with $F\supsetneqq I_0\supset J$ and $\mathsf{P}_{F,J}^\omega\neq 0$, then*

$$F=\pi_{\mathcal{F}}^{(\ell)}I_0\ \textit{for some}\ 0\leq\ell\leq\boldsymbol{\tau}.$$

As a consequence we have the bounded overlap,

$$\#\left\{F\in\mathcal{F}:J\subset I_0\subsetneqq F\ \textit{for some}\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}\left(F\right)\ \textit{with}\ \mathsf{P}_{F,J}^\omega\neq 0\right\}\leq\boldsymbol{\tau}.$$

Proof. Indeed, if $J'\in\mathcal{C}_{\pi_{\mathcal{F}}^{(\ell)}I_0}^{\text{good},\boldsymbol{\tau}-\text{shift}}$ for some $\ell>\boldsymbol{\tau}$, then either $J'\cap\pi_{\mathcal{F}}^{(0)}I_0=\emptyset$ or $J'\supset\pi_{\mathcal{F}}^{(0)}I_0$. Since $J\subset I_0\subset\pi_{\mathcal{F}}^{(0)}I_0$, we cannot have J' contained in J, and this shows that $\mathsf{P}_{\pi_{\mathcal{F}}^{(\ell)}I_0,J}^\omega=0$. □

Finally we record the only places in the proof where the *refined* quasienergy conditions are used. This lemma will be used in bounding both of the local Poisson testing conditions. Recall that $\mathcal{A}\Omega\mathcal{D}$ consists of all alternate $\Omega\mathcal{D}$-dyadic quasicubes where K is alternate dyadic if it is a union of 2^n $\Omega\mathcal{D}$-dyadic quasicubes K' with $\ell\left(K'\right)=\frac{1}{2}\ell\left(K\right)$.

Remark 12. *The following lemma is another of the key results on the way to the proof of our theorem, and is an analogue of the corresponding lemma from* [SaShUr5]*, but with the right hand side involving only the plugged energy constants and the energy Muckenhoupt constants.*

Lemma 16. *Let $\Omega\mathcal{D},\mathcal{F}\subset\Omega\mathcal{D}$ be quasigrids and $\left\{\mathsf{P}_{F,J}^{\omega}\right\}_{\substack{F\in\mathcal{F}\\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}}$ be as above with J,F in the dyadic quasigrid $\Omega\mathcal{D}$. For any alternate quasicube $I\in\mathcal{A}\Omega\mathcal{D}$ define*

$$B(I)\equiv\sum_{F\in\mathcal{F}:\ F\supsetneqq I'\ \text{for some}\ I'\in\mathfrak{C}(I)}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset I}\left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 . \tag{9.5}$$

Then

$$B(I)\lesssim\boldsymbol{\tau}\left(\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2+A_2^{\alpha,\mathrm{energy}}\right)|I|_{\sigma} . \tag{9.6}$$

Proof. We first prove the bound (9.6) for $B(I)$ ignoring for the moment the possible case when $J=I$ in the sum defining $B(I)$. So suppose that $I\in\mathcal{A}\Omega\mathcal{D}$ is an alternate $\Omega\mathcal{D}$-dyadic quasicube. Define

$$\Lambda^*(I)\equiv\left\{J\subsetneqq I: J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)\ \text{ for some } F\supsetneqq I',\ I'\in\mathfrak{C}(I)\ \text{ with } \mathsf{P}_{F,J}^{\omega}\neq 0\right\},$$

and pigeonhole this collection as $\Lambda^*(I)=\bigcup_{I_0'\in\mathfrak{C}(I)}\Lambda(I')$, where for each $I'\in\mathfrak{C}(I)$ we define

$$\Lambda(I')\equiv\left\{J\subset I': J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)\ \text{ for some } F\supsetneqq I' \text{ with } \mathsf{P}_{F,J}^{\omega}\neq 0\right\}.$$

By Lemma 15 we may further pigeonhole (possibly with some duplication) the quasicubes J in $\Lambda(I')$ as follows:

$$\Lambda(I')\subset\bigcup_{\ell=0}^{\boldsymbol{\tau}}\Lambda_{\ell}(I');\qquad\Lambda_{\ell}(I')\equiv\left\{J\subset I': J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(\pi_{\mathcal{F}}^{\ell}I'\right)\ \text{ with } \mathsf{P}_{\pi_{\mathcal{F}}^{\ell}I',J}^{\omega}\neq 0\right\},$$

since every $F\supsetneqq I'$ is of the form $\pi_{\mathcal{F}}^{\ell}I'$ for some $\ell\geq 0$. Altogether then, we have pigeonholed $\Lambda^*(I)$ as

$$\Lambda^*(I)=\bigcup_{I'\in\mathfrak{C}(I)}\bigcup_{\ell=0}^{\boldsymbol{\tau}}\Lambda_{\ell}(I').$$

Now fix $I'\in\mathfrak{C}(I)$ and $0\leq\ell\leq\boldsymbol{\tau}$, and for each J in $\Lambda_{\ell}(I')$, note that *either* J must contain some $K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I')$ *or* $J\subset K$ for some $K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I')$ (or both if equality); define

$$\begin{aligned}\Lambda_{\ell}(I') &= \Lambda_{\ell}^{\mathrm{big}}(I')\cup\Lambda_{\ell}^{\mathrm{small}}(I');\\ \Lambda_{\ell}^{\mathrm{small}}(I') &\equiv \left\{J\in\Lambda_{\ell}(I'): J\subset K \text{ for some } K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I')\right\},\end{aligned}$$

and we make the corresponding decomposition of $B(I)$ (again with possible duplication);

$$\begin{aligned}B(I) &= B^{\mathrm{big}}(I)+B^{\mathrm{small}}(I);\\ B^{\mathrm{big}\,/\,\mathrm{small}}(I) &\equiv \sum_{I'\in\mathfrak{C}(I)}\sum_{\ell=0}^{\boldsymbol{\tau}}\sum_{J\in\Lambda_{\ell}^{\mathrm{big}\,/\,\mathrm{small}}(I')}\left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_{\pi_{\mathcal{F}}^{\ell}I',J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 .\end{aligned}$$

Turning first to $B^{\mathrm{small}}(I)$, we define $\sigma(\ell)$ by $\pi_{\mathcal{F}}^{(\ell)}(I')=\pi_{\Omega\mathcal{D}}^{(\sigma(\ell))}(I')$, so that $\Lambda_{\ell}^{\mathrm{small}}(I')\subset\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}^{\sigma(\ell)}(I)$, and we obtain

$$\begin{aligned}B^{\mathrm{small}}(I) &\leq \sum_{I'\in\mathfrak{C}(I)}\sum_{\ell=0}^{\boldsymbol{\tau}}\sum_{J\in\Lambda_{\ell}^{\mathrm{small}}(I')}\left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_J^{\mathrm{good},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\\ &\leq \sum_{I'\in\mathfrak{C}(I)}\sum_{\ell=0}^{\boldsymbol{\tau}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}^{\sigma(\ell)}(I)}\left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_J^{\mathrm{good},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\\ &\lesssim 2^n\boldsymbol{\tau}\left(\mathcal{E}_{\alpha}^{\mathrm{refined\,plug}}\right)^2|I|_{\sigma} .\end{aligned} \tag{9.7}$$

This is the only point in the proof of Theorem 1 that a refined quasienergy constant is used.

Turning now to the more delicate term $B^{\mathrm{big}}(I)$, we write for $J \in \Lambda_\ell^{\mathrm{big}}(I')$,

$$\begin{aligned}\left\|\mathsf{P}_J^{\mathrm{good},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 &= \sum_{J'\subset J:\ J'\text{ good}} \left\|\triangle_{J'}^\omega \mathbf{x}\right\|_{L^2(\omega)}^2 \\ &\leq \sum_{J'\in\mathcal{N}_{\mathbf{r}}(I'):\ J'\subset J} \left\|\triangle_{J'}^\omega \mathbf{x}\right\|_{L^2(\omega)}^2 + \sum_{K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I'):\ K\subset J} \left\|\mathsf{P}_K^{\mathrm{good},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 ,\end{aligned}$$

where for any I, we define $\mathcal{N}_{\mathbf{r}}(I) \equiv \{J' \subset I : \ell(J') \geq 2^{-\mathbf{r}}\ell(I)\}$ to be the set of r-near quasicubes in I. Then we estimate

$$\begin{aligned} B^{\mathrm{big}}(I) &= \sum_{I'\in\mathfrak{C}(I)}\sum_{\ell=0}^{\boldsymbol{\tau}}\sum_{J\in\Lambda_\ell^{\mathrm{big}}(I')} \left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_{\pi_{\mathcal{F}}^\ell I',J}^\omega \mathbf{x}\right\|_{L^2(\omega)}^2 \\ &\leq \sum_{I'\in\mathfrak{C}(I)}\sum_{\ell=0}^{\boldsymbol{\tau}}\sum_{J\in\Lambda_\ell^{\mathrm{big}}(I')} \left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_J^{\mathrm{good},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\ &= \sum_{I'\in\mathfrak{C}(I)}\sum_{\ell=0}^{\boldsymbol{\tau}}\sum_{J\in\Lambda_\ell^{\mathrm{big}}(I')} \left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2 \sum_{J'\in\mathcal{N}_{\mathbf{r}}(I'):\ J'\subset J} \left\|\triangle_{J'}^\omega \mathbf{x}\right\|_{L^2(\omega)}^2 \\ &\quad + \sum_{I'\in\mathfrak{C}(I)}\sum_{\ell=0}^{\boldsymbol{\tau}}\sum_{J\in\Lambda_\ell^{\mathrm{big}}(I')} \left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2 \sum_{K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I'):\ K\subset J} \left\|\mathsf{P}_K^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\ &\equiv B_1^{\mathrm{big}}(I) + B_2^{\mathrm{big}}(I) .\end{aligned}$$

Now using that the quasicubes J in each $\Lambda_\ell^{\mathrm{big}}(I')$ that arise in the term $B_1^{\mathrm{big}}(I)$ are both $\mathbf{r}$-nearby in I' and pairwise disjoint, and that there are altogether only $C2^{n\mathbf{r}}$ quasicubes in $\mathcal{N}_{\mathbf{r}}(I')$, we have

(9.8)

$$\begin{aligned} B_1^{\mathrm{big}}(I) &= \sum_{I'\in\mathfrak{C}(I)}\sum_{\ell=0}^{\boldsymbol{\tau}}\sum_{J\in\Lambda_\ell^{\mathrm{big}}(I')\cap\mathcal{N}_{\mathbf{r}}(I')} \left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2 \sum_{J'\in\mathcal{N}_{\mathbf{r}}(I'):\ J'\subset J} \left\|\triangle_{J'}^\omega \mathbf{x}\right\|_{L^2(\omega)}^2 \\ &\lesssim \sum_{I'\in\mathfrak{C}(I)}\sum_{\ell=0}^{\boldsymbol{\tau}}\sum_{J\in\Lambda_\ell^{\mathrm{big}}(I')\cap\mathcal{N}_{\mathbf{r}}(I')} \left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_J^\omega \mathbf{x}\right\|_{L^2(\omega)}^2 \\ &\lesssim \boldsymbol{\tau}2^{n\mathbf{r}} \sup_{I'\in\mathfrak{C}(I)}\sup_{J\in\mathcal{N}_{\mathbf{r}}(I')} \left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_J^\omega \mathbf{x}\right\|_{L^2(\omega)}^2 \\ &\lesssim \boldsymbol{\tau}2^{n\mathbf{r}} \left(\frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}}\right)^2 \left\|\mathsf{P}_I^\omega \frac{\mathbf{x}}{\ell(I)}\right\|_{L^2(\omega)}^2 \lesssim \boldsymbol{\tau}2^{n\mathbf{r}} A_2^{\alpha,\mathrm{energy}} |I|_\sigma .\end{aligned}$$

At this point we can estimate the missing case $J = I$ in the same way, namely

$$\begin{aligned} &\sum_{F\in\mathcal{F}:\ I\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)} \left(\frac{\mathrm{P}^\alpha(I,\mathbf{1}_I\sigma)}{|I|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_{F,I}^\omega \mathbf{x}\right\|_{L^2(\omega)}^2 \\ &\lesssim \boldsymbol{\tau}\left(\frac{\mathrm{P}^\alpha(I,\mathbf{1}_I\sigma)}{|I|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_I^\omega \mathbf{x}\right\|_{L^2(\omega)}^2 \lesssim \boldsymbol{\tau} A_2^{\alpha,\mathrm{energy}} |I|_\sigma .\end{aligned}$$

Since $\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}} \lesssim \frac{\mathrm{P}^{\alpha}(K,\mathbf{1}_I\sigma)}{|K|^{\frac{1}{n}}}$ for $K \subset J$, and since the quasicubes $J \in \Lambda_{\ell}^{\mathrm{big}}(I')$ are pairwise disjoint, we have

$$\begin{aligned}
B_2^{\mathrm{big}}(I) &= \sum_{I' \in \mathfrak{C}(I)} \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{J \in \Lambda_{\ell}^{\mathrm{big}}(I')} \sum_{K \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I'):\ K \subset J} \left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_I\sigma)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_K^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\
&\lesssim \sum_{I' \in \mathfrak{C}(I)} \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{J \in \Lambda_{\ell}^{\mathrm{big}}(I')} \sum_{K \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I'):\ K \subset J} \left(\frac{\mathrm{P}^{\alpha}(K,\mathbf{1}_I\sigma)}{|K|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_K^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\
&\leq \sum_{I' \in \mathfrak{C}(I)} \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{K \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I')} \left(\frac{\mathrm{P}^{\alpha}(K,\mathbf{1}_I\sigma)}{|K|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_K^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\
&\lesssim \boldsymbol{\tau}\left(\mathcal{E}_{\alpha}^{\mathrm{deep\,plug}}\right)^2 |I|_{\sigma}\ ,
\end{aligned}$$

where the final line follows from the plugged deep energy condition with the trivial outer decomposition $I = \bigcup_{I' \in \mathfrak{C}(I)} I'$. This completes the proof of Lemma 16. $\square$

9.2. **The forward Poisson testing inequality.** Fix $I \in \Omega\mathcal{D}$. We split the integration on the left side of (9.3) into a local and global piece:

$$\int_{\mathbb{R}_+^{n+1}} \mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)^2\, d\overline{\mu} = \int_{\widehat{I}} \mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)^2\, d\overline{\mu} + \int_{\mathbb{R}_+^{n+1}\setminus\widehat{I}} \mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)^2\, d\overline{\mu} \equiv \mathbf{Local}(I) + \mathbf{Global}(I),$$

where more explicitly,

$$\begin{aligned}
&\mathbf{Local}(I) \equiv \int_{\widehat{I}} \left[\mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)(x,t)\right]^2 d\overline{\mu}(x,t)\,; \qquad \overline{\mu} \equiv \frac{1}{t^2}\mu, \qquad (9.9)\\
&\text{i.e. } \overline{\mu} \equiv \sum_{J \in \Omega\mathcal{D}} \frac{1}{\ell(J)^2} \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)} \left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \cdot \delta_{(c_J,\ell(J))}.
\end{aligned}$$

Here is a brief schematic diagram of the decompositions, with bounds in $\Box$, used in this subsection:

$$\boxed{\begin{array}{ccc}
\mathbf{Local}(I) & & \\
\downarrow & & \\
\mathbf{Local}^{\mathrm{plug}}(I) & + & \mathbf{Local}^{\mathrm{hole}}(I) \\
& & \boxed{\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}\right)^2} \\
\downarrow & & \\
\downarrow & & \\
A & + & B \\
\boxed{\left(\mathcal{E}_{\alpha}^{\mathrm{deep\,plug}}\right)^2} & & \boxed{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2 + A_2^{\alpha,\mathrm{energy}}}
\end{array}}$$

and

$$\boxed{\begin{array}{ccccccc}
\mathbf{Global}(I) & & & & & & \\
\downarrow & & & & & & \\
A & + & B & + & C & + & D \\
\boxed{A_2^{\alpha}} & & \boxed{A_2^{\alpha} + A_2^{\alpha,\mathrm{energy}}} & & \boxed{\mathcal{A}_2^{\alpha,*}} & & \boxed{\mathcal{A}_2^{\alpha,*} + A_2^{\alpha,\mathrm{energy}} + A_2^{\alpha,\mathrm{punct}}}
\end{array}}\ .$$

An important consequence of the fact that I and J lie in the same quasigrid $\Omega\mathcal{D} = \Omega\mathcal{D}^{\omega}$, is that

$$(c(J), \ell(J)) \in \widehat{I} \textbf{ if and only if } J \subset I. \qquad (9.10)$$

We thus have

$$\begin{aligned}
&\mathbf{Local}\left(I\right)=\int_{\widehat{I}}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(x,t\right)^{2}d\overline{\mu}\left(x,t\right)\\
=&\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(F):\ J\subset I}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(c_{J},\left|J\right|^{\frac{1}{n}}\right)^{2}\left\|\mathsf{P}_{F,J}^{\omega}\frac{\mathbf{x}}{\left|J\right|^{\frac{1}{n}}}\right\|_{L^{2}(\omega)}^{2}\\
\approx&\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(F):\ J\subset I}\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I}\sigma\right)^{2}\|\mathsf{P}_{F,J}^{\omega}\frac{\mathbf{x}}{\left|J\right|^{\frac{1}{n}}}\|_{L^{2}(\omega)}^{2}\\
\lesssim&\ \mathbf{Local}^{\mathrm{plug}}\left(I\right)+\mathbf{Local}^{\mathrm{hole}}\left(I\right),
\end{aligned}$$

where the 'plugged' local sum $\mathbf{Local}^{\mathrm{plug}}\left(I\right)$ is given by

$$\begin{aligned}
&\mathbf{Local}^{\mathrm{plug}}\left(I\right)\equiv\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(F):\ J\subset I}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{F\cap I}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}^{2}\\
&=\left\{\sum_{F\in\mathcal{F}:\ F\subset I}+\sum_{F\in\mathcal{F}:\ F\supsetneqq I}\right\}\sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(F):\ J\subset I}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{F\cap I}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}^{2}\\
&=A+B.
\end{aligned}$$

Then a *trivial* application of the deep quasienergy condition (where 'trivial' means that the outer decomposition is just a single quasicube) gives

$$\begin{aligned}
A&\leq\sum_{F\in\mathcal{F}:\ F\subset I}\sum_{J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(F)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{F}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}^{2}\\
&\leq\sum_{F\in\mathcal{F}:\ F\subset I}\left(\mathcal{E}_{\alpha}^{\mathrm{deep\,plug}}\right)^{2}\left|F\right|_{\sigma}\lesssim\left(\mathcal{E}_{\alpha}^{\mathrm{deep\,plug}}\right)^{2}\left|I\right|_{\sigma}\ ,
\end{aligned}$$

since $\left\|\mathsf{P}_{F,J}^{\omega}x\right\|_{L^{2}(\omega)}^{2}\leq\left\|\mathsf{P}_{J}^{\mathrm{good},\omega}\mathbf{x}\right\|_{L^{2}(\omega)}^{2}$, where we recall that the quasienergy constant $\mathcal{E}_{\alpha}^{\mathrm{deep\,plug}}$ is defined in (3.9). We also used here that the stopping quasicubes $\mathcal{F}$ satisfy a σ-Carleson measure estimate,

$$\sum_{F\in\mathcal{F}:\ F\subset F_{0}}\left|F\right|_{\sigma}\lesssim\left|F_{0}\right|_{\sigma}.$$

Lemma 16 applies to the remaining term B to obtain the bound

$$B\lesssim\boldsymbol{\tau}\left(\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^{2}+A_{2}^{\alpha,\mathrm{energy}}\right)\left|I\right|_{\sigma}\ .$$

It remains then to show the inequality with 'holes', where the support of σ is restricted to the complement of the quasicube F. Thus for $J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(F\right)$ we may use $I\setminus F$ in the argument of the Poisson integral. We consider

$$\mathbf{Local}^{\mathrm{hole}}\left(I\right)=\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset I}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I\setminus F}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}^{2}\ .$$

Lemma 17. *We have*

$$\mathbf{Local}^{\mathrm{hole}}\left(I\right)\lesssim\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}\right)^{2}\left|I\right|_{\sigma}\ . \tag{9.11}$$

Proof. Fix $I\in\Omega\mathcal{D}$ and define

$$\mathcal{F}_{I}\equiv\left\{F\in\mathcal{F}:F\subset I\right\}\cup\left\{I\right\},$$

and denote by πF, for this proof only, the parent of F in the tree $\mathcal{F}_{I}$. We estimate

$$S\equiv\sum_{F\in\mathcal{F}_{I}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset I}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I\setminus F}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}^{2}$$

by

$$\begin{aligned} S &= \sum_{F\in\mathcal{F}_I}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset I}\ \sum_{F'\in\mathcal{F}:\ F\subset F'\subsetneqq I}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}^\omega_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)} \\ &= \sum_{F'\in\mathcal{F}_I}\sum_{F\in\mathcal{F}:\ F\subset F'}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset I}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}^\omega_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)} \\ &= \sum_{F'\in\mathcal{F}_I}\sum_{K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F')}\ \sum_{F\in\mathcal{F}:\ F\subset F'}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset I}\left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}^\omega_{F,J\cap K}\mathbf{x}\right\|^2_{L^2(\omega)} \\ &\lesssim \sum_{F'\in\mathcal{F}_I}\sum_{K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F')}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2\sum_{F\in\mathcal{F}:\ F\subset F'}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset I}\left\|\mathsf{P}^\omega_{F,J\cap K}\mathbf{x}\right\|^2_{L^2(\omega)}, \end{aligned}$$

where in the third line we have used that each J' appearing in $\mathsf{P}^\omega_{F,J}$ occurs in one of the $\mathsf{P}^\omega_{F,J\cap K}$ by goodness, and where in the fourth line we have used the Poisson inequalities in Lemma 14. We now invoke

$$\sum_{F\in\mathcal{F}:\ F\subset F'}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset I}\left\|\mathsf{P}^\omega_{F,J\cap K}\mathbf{x}\right\|^2_{L^2(\omega)}\lesssim\boldsymbol{\tau}\left\|\widehat{\mathsf{P}}^\omega_{F',K}\mathbf{x}\right\|^2_{L^2(\omega)}\ ,$$

where for $K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F')$,

$$\begin{aligned} \widehat{\mathsf{P}}^\omega_{F',K} &\equiv \sum_{J'\subset J\cap K:\ J'\in\mathcal{C}^*_{F';I}}\triangle^\omega_{J'}\ , \\ \text{where } \mathcal{C}^*_{F';I} &\equiv \bigcup_{F\in\mathcal{F}:\ F\subset F'}\ \bigcup_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset I}\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}\ . \end{aligned}$$

Now denote by $d(F,F')\equiv d_{\mathcal{F}_I}(F,F')$ the distance from F to F' in the tree $\mathcal{F}_I$, and denote by $d(F)\equiv d_{\mathcal{F}_I}(F,I)$ the distance of F from the root I. Since the collection $\mathcal{F}$ satisfies a Carleson condition, namely $\sum_{F\in\mathcal{F}_I}|F\cap I'|_\sigma\le C|I'|_\sigma$ for all I', we have geometric decay in generations:

$$\sum_{F\in\mathcal{F}_I:\ d(F)=k}|F|_\sigma\lesssim 2^{-\delta k}|I|_\sigma\ ,\qquad k\ge 0. \tag{9.12}$$

Indeed, with $m>2C$ we have

$$\sum_{F\in\mathcal{F}_I:\ F\subset F'\text{ and }d(F,F')=m}|F'\cap I'|_\sigma<\frac{1}{2}|I'|_\sigma\ , \tag{9.13}$$

since otherwise

$$\sum_{F\in\mathcal{F}_I:\ F\subset F'\text{ and }d(F,F')\le m}|F'\cap I'|_\sigma\ge m\frac{1}{2}|I'|_\sigma\ ,$$

a contradiction. Now iterate (9.13) to obtain (9.12). Thus we can write

$$\begin{aligned} S &\lesssim \sum_{F'\in\mathcal{F}_I}\sum_{K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F')}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2\left\|\widehat{\mathsf{P}}^\omega_{F',K}\mathbf{x}\right\|^2_{L^2(\omega)} \\ &= \sum_{k=1}^\infty\sum_{F'\in\mathcal{F}_I:\ d(F')=k}\ \sum_{K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F')}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2\left\|\widehat{\mathsf{P}}^\omega_{F',K}\mathbf{x}\right\|^2_{L^2(\omega)}\equiv\sum_{k=1}^\infty A_k. \end{aligned}$$

Now we have $\left\|\widehat{\mathsf{P}}^{\omega}_{F',K}\mathbf{x}\right\|^2_{L^2(\omega)} \leq \left\|\mathsf{P}^{\mathrm{good},\omega}_{K}\mathbf{x}\right\|^2_{L^2(\omega)}$, and hence by the deep energy condition,

$$\begin{aligned} A_k &= \sum_{F'\in\mathcal{F}_I:\ d(F')=k}\ \sum_{K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F')} \left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{\pi_{\mathcal{F}}F'\setminus F'}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \left\|\widehat{\mathsf{P}}^{\omega}_{F',K}\mathbf{x}\right\|^2_{L^2(\omega)} \\ &\lesssim \left(\mathcal{E}^{\mathrm{deep}}_{\alpha}\right)^2 \sum_{F''\in\mathcal{F}_I:\ d(F'')=k-1} |F''|_{\sigma} \lesssim \left(\mathcal{E}^{\mathrm{deep}}_{\alpha}\right)^2 2^{-\delta k}|I|_{\sigma}\ , \end{aligned}$$

where we have applied the deep energy condition for each $F''\in\mathcal{F}_I$ with $d(F'')=k-1$ to obtain

$$\sum_{F'\in\mathcal{F}_I:\ \pi F'=F''}\ \sum_{K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F')} \left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{F''\setminus F'}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \left\|\widehat{\mathsf{P}}^{\omega}_{F',K}\mathbf{x}\right\|^2_{L^2(\omega)} \leq \left(\mathcal{E}^{\mathrm{deep}}_{\alpha}\right)^2 |F''|_{\sigma}\ . \tag{9.14}$$

Finally then we obtain

$$S \lesssim \sum_{k=1}^{\infty} \left(\mathcal{E}^{\mathrm{deep}}_{\alpha}\right)^2 2^{-\delta k}|I|_{\sigma} \lesssim \left(\mathcal{E}^{\mathrm{deep}}_{\alpha}\right)^2 |I|_{\sigma}\ ,$$

which is (9.11). $\square$

Altogether we have now proved the estimate $\mathbf{Local}(I) \lesssim \left(\left(\mathcal{E}^{\mathrm{plug}}_{\alpha}\right)^2 + A_2^{\alpha,\mathrm{energy}}\right)|I|_{\sigma}$ when $I\in\Omega\mathcal{D}$, i.e. for every dyadic quasicube $L\in\Omega\mathcal{D}$,

(9.15)

$$\begin{aligned} \mathbf{Local}(L) &\approx \sum_{F\in\mathcal{F}}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset L} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{L}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)} \\ &\lesssim \left(\left(\mathcal{E}^{\mathrm{plug}}_{\alpha}\right)^2 + A_2^{\alpha,\mathrm{energy}}\right)|L|_{\sigma}, \quad L\in\Omega\mathcal{D}. \end{aligned}$$

9.2.1. *The alternate local estimate.* For future use, we prove a strengthening of the local estimate $\mathbf{Local}(L)$ to *alternate* quasicubes $M\in\mathcal{A}\Omega\mathcal{D}$.

Lemma 18. *With notation as above and $M\in\mathcal{A}\Omega\mathcal{D}$ an alternate quasicube, we have*

(9.16)

$$\begin{aligned} \mathbf{Local}(M) &\equiv \sum_{F\in\mathcal{F}}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset M} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{M}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)} \\ &\lesssim \boldsymbol{\tau}\left(\left(\mathcal{E}^{\mathrm{plug}}_{\alpha}\right)^2 + A_2^{\alpha,\mathrm{energy}}\right)|M|_{\sigma}, \quad M\in\mathcal{A}\Omega\mathcal{D}. \end{aligned}$$

Proof. We prove (9.16) by repeating the above proof of (9.15) and noting the points requiring change. First we decompose

$$\mathbf{Local}(M) \lesssim \mathbf{Local}^{\mathrm{plug}}(M) + \mathbf{Local}^{\mathrm{hole}}(M) + \mathbf{Local}^{\mathrm{offset}}(M)$$

where $\mathbf{Local}^{\mathrm{plug}}(M)$ and $\mathbf{Local}^{\mathrm{hole}}(M)$ are analogous to $\mathbf{Local}^{\mathrm{plug}}(I)$ and $\mathbf{Local}^{\mathrm{hole}}(I)$ above, and where $\mathbf{Local}^{\mathrm{offset}}(M)$ is an additional term arising because $M\setminus F$ need not be empty when $M\cap F\neq\emptyset$ and F is not contained in M:

$$\begin{aligned} \mathbf{Local}^{\mathrm{plug}}(M) &\equiv \sum_{F\in\mathcal{F}}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset M} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{M\cap F}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)}\ , \\ \mathbf{Local}^{\mathrm{hole}}(M) &\equiv \sum_{F\in\mathcal{F}:\ F\subset M}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset M} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{M\setminus F}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)}\ , \\ \mathbf{Local}^{\mathrm{offset}}(M) &\equiv \sum_{F\in\mathcal{F}:\ F\not\subset M}\ \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset M} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{M\setminus F}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)}\ . \end{aligned}$$

We have

$$\mathbf{Local}^{\mathrm{plug}}(M)=\left\{\sum_{F\in\mathcal{F}:\ F\subset\text{ some }M'\in\mathfrak{C}_{\Omega\mathcal{D}}(M)}+\sum_{F\in\mathcal{F}:\ F\supsetneqq\text{ some }M'\in\mathfrak{C}_{\Omega\mathcal{D}}(M)}\right\}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset M}$$
$$\times\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{F\cap M}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^{2}\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}^{2}$$
$$=A+B.$$

Term A satisfies

$$A\lesssim\left(\mathcal{E}_{\alpha}^{\mathrm{deep\,plug}}\right)^{2}|M|_{\sigma}\ ,$$

just as above using $\left\|\mathsf{P}_{F,J}^{\omega}x\right\|_{L^{2}(\omega)}^{2}\leq\left\|\mathsf{P}_{J}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}^{2}$, and the fact that the stopping quasicubes $\mathcal{F}$ satisfy a σ-Carleson measure estimate,

$$\sum_{F\in\mathcal{F}:\ F\subset M}|F|_{\sigma}\lesssim|M|_{\sigma}\,.$$

Term B is handled directly by Lemma 16 with the alternate quasicube $I=M$ to obtain

$$B\lesssim\left(\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^{2}+A_{2}^{\alpha,\mathrm{energy}}\right)|M|_{\sigma}\ .$$

To extend Lemma 17 to alternate quasicubes $M\in\mathcal{A}\Omega\mathcal{D}$, we define

$$\mathcal{F}_{M}\equiv\{F\in\mathcal{F}:F\subset M\}\cup\{M\}\,,$$

and follow along the proof there with only trivial changes. The analogue of (9.14) is now

$$\sum_{F'\in\mathcal{F}_{M}:\ \pi F'=F''}\ \sum_{K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F')}\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{F''\setminus F'}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^{2}\left\|\widehat{\mathsf{P}}_{F',K}^{\omega}\mathbf{x}\right\|_{L^{2}(\omega)}^{2}\leq\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}\right)^{2}|F''|_{\sigma}\ ,$$

the only change being that $\mathcal{F}_{M}$ now appears in place of $\mathcal{F}_{I}$, so that the deep energy condition still applies. We conclude that

$$\mathbf{Local}^{\mathrm{hole}}(M)\lesssim\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}\right)^{2}|M|_{\sigma}\ .$$

Finally, the additional term $\mathbf{Local}^{\mathrm{offset}}(M)$ is handled directly by Lemma 16, and this completes the proof of the estimate (9.16) in Lemma 18. □

9.2.2. *The global estimate.* Now we turn to proving the following estimate for the global part of the first testing condition (9.3):

$$\mathbf{Global}(I)=\int_{\mathbb{R}_{+}^{n+1}\setminus\widehat{I}}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)^{2}d\overline{\mu}\lesssim\left(\mathcal{A}_{2}^{\alpha,*}+A_{2}^{\alpha,\mathrm{energy}}+A_{2}^{\alpha,\mathrm{punct}}\right)|I|_{\sigma}\,.$$

We begin by decomposing the integral on the right into four pieces. As a particular consequence of Lemma 15, we note that given J, there are at most a fixed number $\boldsymbol{\tau}$ of $F\in\mathcal{F}$ such that $J\in\mathcal{M}_{\mathbf{r}-\mathrm{deep}}(F)$. We have:

$$\int_{\mathbb{R}_{+}^{n+1}\setminus\widehat{I}}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)^{2}d\mu\leq\sum_{J:\ (c_{J},\ell(J))\in\mathbb{R}_{+}^{n+1}\setminus\widehat{I}}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(c_{J},\ell(J)\right)^{2}\sum_{\substack{F\in\mathcal{F}\\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}}\left\|\mathsf{P}_{F,J}^{\omega}\frac{\mathbf{x}}{|J|^{\frac{1}{n}}}\right\|_{L^{2}(\omega)}^{2}$$
$$=\left\{\sum_{\substack{J\cap3I=\emptyset\\ \ell(J)\leq\ell(I)}}+\sum_{J\subset3I\setminus I}+\sum_{\substack{J\cap I=\emptyset\\ \ell(J)>\ell(I)}}+\sum_{J\supsetneqq I}\right\}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(c_{J},\ell(J)\right)^{2}\sum_{\substack{F\in\mathcal{F}:\\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}}\left\|\mathsf{P}_{F,J}^{\omega}\frac{\mathbf{x}}{|J|^{\frac{1}{n}}}\right\|_{L^{2}(\omega)}^{2}$$
$$=A+B+C+D.$$

We further decompose term A according to the length of J and its distance from I, and then use Lemma 15 with $I_0 = J$ to obtain:

$$\begin{aligned}
A &\lesssim \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\sum_{\substack{J\subset 3^{k+1}I\setminus 3^k I\\ \ell(J)=2^{-m}\ell(I)}}\left(\frac{2^{-m}\left|I\right|^{\frac{1}{n}}}{\operatorname{quasidist}(J,I)^{n+1-\alpha}}\left|I\right|_{\sigma}\right)^{2}\boldsymbol{\tau}\left|J\right|_{\omega}\\
&\lesssim \sum_{m=0}^{\infty}2^{-2m}\sum_{k=1}^{\infty}\frac{\left|I\right|^{\frac{2}{n}}\left|I\right|_{\sigma}\left|3^{k+1}I\setminus 3^k I\right|_{\omega}}{\left|3^k I\right|^{2\left(1+\frac{1}{n}-\frac{\alpha}{n}\right)}}\left|I\right|_{\sigma}\\
&\lesssim \sum_{m=0}^{\infty}2^{-2m}\sum_{k=1}^{\infty}3^{-2k}\left\{\frac{\left|3^{k+1}I\setminus 3^k I\right|_{\omega}\left|3^k I\right|_{\sigma}}{\left|3^k I\right|^{2\left(1-\frac{\alpha}{n}\right)}}\right\}\left|I\right|_{\sigma}\lesssim A_2^{\alpha}\left|I\right|_{\sigma},
\end{aligned}$$

where the offset Muckenhoupt constant A_2^{α} applies because $3^{k+1}I$ has only three times the side length of $3^k I$.

For term B we first dispose of the nearby sum B_{nearby} that consists of the sum over those J which satisfy in addition $2^{-\mathbf{r}}\ell(I)\leq\ell(J)\leq\ell(I)$. But it is a straightforward task to bound B_{nearby} by $CA_2^{\alpha,\text{energy}}\left|I\right|_{\sigma}$ as there are at most $2^{n(\mathbf{r}+1)}$ such quasicubes J. In order to bound $B-B_{\text{nearby}}$, let

$$\mathcal{J}^{*}\equiv\bigcup_{F\in\mathcal{F}}\bigcup_{\substack{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F)\\ J\subset 3I\setminus I\text{ and }\ell(J)\leq 2^{-\mathbf{r}}\ell(I)}}\left\{K\in\mathcal{C}_F^{\text{good},\boldsymbol{\tau}-\text{shift}}:K\subset J\right\},$$

which is the union of all quasicubes K for which the projection $\triangle_K^{\omega}$ occurs in one of the projections $\mathsf{P}_{F,J}^{\omega}$ with $\ell(J)<2^{-\mathbf{r}}\ell(I)$ in term B. We further decompose term B according to the length of J and use the fractional Poisson inequality (7.13) in Lemma 13 on the neighbour I' of I containing K,

$$\mathrm{P}^{\alpha}\left(K,\mathbf{1}_I\sigma\right)^{2}\lesssim\left(\frac{\ell(K)}{\ell(I)}\right)^{2-2(n+1-\alpha)\varepsilon}\mathrm{P}^{\alpha}\left(I,\mathbf{1}_I\sigma\right)^{2},\qquad K\in\mathcal{J}^{*},K\subset 3I\setminus I,$$

where we have used that $\mathrm{P}^{\alpha}(I',\mathbf{1}_I\sigma)\approx\mathrm{P}^{\alpha}(I,\mathbf{1}_I\sigma)$ and that the quasicubes $K\in\mathcal{J}^{*}$ are good and have side length at most $2^{-\mathbf{r}}\ell(I)$. We then obtain from Lemma 14 and Lemma 15 with $I_0=J$,

$$\begin{aligned}
B-B_{\text{nearby}} &\approx \sum_{J\subset 3I\setminus I}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_I\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\sum_{\substack{F\in\mathcal{F}:\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F)\\ \ell(J)\leq 2^{-\mathbf{r}}\ell(I)}}\left\|\mathsf{P}_{F,J}^{\omega}x\right\|_{L^2(\omega)}^{2}\\
&\lesssim \sum_{K\in\mathcal{J}^{*}}\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_I\sigma\right)}{\left|K\right|^{\frac{1}{n}}}\right)^{2}\boldsymbol{\tau}\ \left\|\triangle_K^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^{2}\\
&\lesssim \boldsymbol{\tau}\sum_{m=\mathbf{r}}^{\infty}\sum_{\substack{K\subset 3I\setminus I\\ \ell(K)=2^{-m}\ell(I)}}\left(2^{-m}\right)^{2-2(n+1-\alpha)\varepsilon}\mathrm{P}^{\alpha}\left(I,\mathbf{1}_I\sigma\right)^{2}\left|K\right|_{\omega}\\
&\lesssim \boldsymbol{\tau}\sum_{m=\mathbf{r}}^{\infty}\left(2^{-m}\right)^{2-2(n+1-\alpha)\varepsilon}\left(\frac{\left|I\right|_{\sigma}}{\left|I\right|^{1-\frac{\alpha}{n}}}\right)^{2}\sum_{\substack{K\subset 3I\setminus I\\ \ell(K)=2^{-m}\ell(I)}}\left|K\right|_{\omega}\\
&\lesssim \boldsymbol{\tau}\sum_{m=\mathbf{r}}^{\infty}\left(2^{-m}\right)^{2-2(n+1-\alpha)\varepsilon}\frac{\left|I\right|_{\sigma}\left|3I\setminus I\right|_{\omega}}{\left|3I\right|^{2\left(1-\frac{\alpha}{n}\right)}}\left|I\right|_{\sigma}\lesssim\boldsymbol{\tau}A_2^{\alpha}\left|I\right|_{\sigma}\ .
\end{aligned}$$

For term C we will have to group the quasicubes J into blocks B_i, and then exploit Lemma 15. We first split the sum according to whether or not I intersects the triple of J:

$$\begin{aligned} C &\approx \left\{ \sum_{\substack{J:\ I\cap 3J=\emptyset \\ \ell(J)>\ell(I)}} + \sum_{\substack{J:\ I\subset 3J\setminus J \\ \ell(J)>\ell(I)}} \right\} \left(\frac{|J|^{\frac{1}{n}}}{\left(|J|^{\frac{1}{n}}+\operatorname{quasidist}(J,I)\right)^{n+1-\alpha}} |I|_\sigma \right)^2 \sum_{\substack{F\in\mathcal{F}: \\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \left\| \mathsf{P}^\omega_{F,J} \frac{\mathbf{x}}{|J|^{\frac{1}{n}}} \right\|^2_{L^2(\omega)} \\ &= C_1 + C_2. \end{aligned}$$

We first consider C_1. Let $\mathcal{M}$ be the maximal dyadic quasicubes in $\{Q : 3Q\cap I=\emptyset\}$, and then let $\{B_i\}_{i=1}^\infty$ be an enumeration of those $Q\in\mathcal{M}$ whose side length is at least $\ell(I)$. Now we further decompose the sum in C_1 by grouping the quasicubes J into the Whitney quasicubes B_i, and then using Lemma 15 with $I_0=J$:

$$\begin{aligned} C_1 &\leq \sum_{i=1}^\infty \sum_{J:\ J\subset B_i\setminus I} \left(\frac{1}{\left(|J|^{\frac{1}{n}}+\operatorname{quasidist}(J,I)\right)^{n+1-\alpha}} |I|_\sigma \right)^2 \sum_{\substack{F\in\mathcal{F}: \\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \left\| \mathsf{P}^\omega_{F,J}\mathbf{x} \right\|^2_{L^2(\omega)} \\ &\lesssim \sum_{i=1}^\infty \left(\frac{1}{\left(|B_i|^{\frac{1}{n}}+\operatorname{quasidist}(B_i,I)\right)^{n+1-\alpha}} |I|_\sigma \right)^2 \sum_{J:\ J\subset B_i\setminus I} \sum_{\substack{F\in\mathcal{F}: \\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \left\| \mathsf{P}^\omega_{F,J}\mathbf{x} \right\|^2_{L^2(\omega)} \\ &\lesssim \sum_{i=1}^\infty \left(\frac{1}{\left(|B_i|^{\frac{1}{n}}+\operatorname{quasidist}(B_i,I)\right)^{n+1-\alpha}} |I|_\sigma \right)^2 \sum_{J:\ J\subset B_i\setminus I} \boldsymbol{\tau}\ |J|^{\frac{2}{n}} |J|_\omega \\ &\lesssim \sum_{i=1}^\infty \left(\frac{1}{\left(|B_i|^{\frac{1}{n}}+\operatorname{quasidist}(B_i,I)\right)^{n+1-\alpha}} |I|_\sigma \right)^2 \boldsymbol{\tau}\ |B_i|^{\frac{2}{n}} |B_i\setminus I|_\omega \\ &\lesssim \boldsymbol{\tau} \left\{ \sum_{i=1}^\infty \frac{|B_i\setminus I|_\omega |I|_\sigma}{|B_i|^{2\left(1-\frac{\alpha}{n}\right)}} \right\} |I|_\sigma\ , \end{aligned}$$

and since $B_i\cap I\neq\emptyset$,

$$\begin{aligned} \sum_{i=1}^\infty \frac{|B_i\setminus I|_\omega |I|_\sigma}{|B_i|^{2\left(1-\frac{\alpha}{n}\right)}} &= \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \sum_{i=1}^\infty \frac{|I|^{1-\frac{\alpha}{n}}}{|B_i|^{2\left(1-\frac{\alpha}{n}\right)}} |B_i\setminus I|_\omega \\ &\approx \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \sum_{i=1}^\infty \int_{B_i\setminus I} \frac{|I|^{1-\frac{\alpha}{n}}}{\operatorname{quasidist}(x,I)^{2(n-\alpha)}} d\omega(x) \\ &\approx \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \sum_{i=1}^\infty \int_{B_i\setminus I} \left(\frac{|I|^{\frac{1}{n}}}{\left[|I|^{\frac{1}{n}}+\operatorname{quasidist}(x,I)\right]^2} \right)^{n-\alpha} d\omega(x) \\ &\leq \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \mathcal{P}^\alpha(I,\mathbf{1}_{I^c}\omega) \leq \mathcal{A}_2^{\alpha,*}, \end{aligned}$$

we obtain

$$C_1 \lesssim \boldsymbol{\tau}\mathcal{A}_2^{\alpha,*} |I|_\sigma\ .$$

Next we turn to estimating term C_2 where the triple of J contains I but J itself does not. Note that there are at most $2^{2n}-2^n$ such quasicubes J of a given side length, at most 2^n-1 in each 'generalized octant'

relative to I. So with this in mind we sum over the quasicubes J according to their lengths to obtain

$$\begin{aligned} C_2 &= \sum_{m=1}^{\infty} \sum_{\substack{J:\ I\subset 3J\setminus J \\ \ell(J)=2^m\ell(I)}} \left(\frac{|J|^{\frac{1}{n}}}{\left(|J|^{\frac{1}{n}}+\operatorname{dist}(J,I)\right)^{n+1-\alpha}} |I|_\sigma \right)^2 \sum_{\substack{F\in\mathcal{F}: \\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \left\| \mathsf{P}_{F,J}^{\omega} \frac{\mathbf{x}}{|J|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2 \\ &\lesssim \sum_{m=1}^{\infty} \left(\frac{|I|_\sigma}{|2^m I|^{1-\frac{\alpha}{n}}} \right)^2 \boldsymbol{\tau}\ |(5\cdot 2^m I)\setminus I|_\omega = \boldsymbol{\tau} \left\{ \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \sum_{m=1}^{\infty} \frac{|I|^{1-\frac{\alpha}{n}} |(5\cdot 2^m I)\setminus I|_\omega}{|2^m I|^{2\left(1-\frac{\alpha}{n}\right)}} \right\} |I|_\sigma \\ &\lesssim \boldsymbol{\tau} \left\{ \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \mathcal{P}^\alpha(I, \mathbf{1}_{I^c}\omega) \right\} |I|_\sigma \le \boldsymbol{\tau} \mathcal{A}_2^{\alpha,*} |I|_\sigma , \end{aligned}$$

since in analogy with the corresponding estimate above,

$$\sum_{m=1}^{\infty} \frac{|I|^{1-\frac{\alpha}{n}} |(5\cdot 2^m I)\setminus I|_\omega}{|2^m I|^{2\left(1-\frac{\alpha}{n}\right)}} = \int \sum_{m=1}^{\infty} \frac{|I|^{1-\frac{\alpha}{n}}}{|2^m I|^{2\left(1-\frac{\alpha}{n}\right)}} \mathbf{1}_{(5\cdot 2^m I)\setminus I}(x)\ d\omega(x) \lesssim \mathcal{P}^\alpha(I, \mathbf{1}_{I^c}\omega) .$$

Finally, we turn to term D. The quasicubes J occurring here are included in the set of ancestors $A_k \equiv \pi_{\Omega\mathcal{D}}^{(k)} I$ of I, $1\le k<\infty$.

$$\begin{aligned} D &= \sum_{k=1}^{\infty} \mathbb{P}^\alpha(\mathbf{1}_I\sigma) \left(c(A_k), |A_k|^{\frac{1}{n}} \right)^2 \sum_{\substack{F\in\mathcal{F}: \\ A_k\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \left\| \mathsf{P}_{F,A_k}^{\omega} \frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2 \\ &= \sum_{k=1}^{\infty} \mathbb{P}^\alpha(\mathbf{1}_I\sigma) \left(c(A_k), |A_k|^{\frac{1}{n}} \right)^2 \sum_{\substack{F\in\mathcal{F}: \\ A_k\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \sum_{J'\in\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}:\ J'\subset A_k\setminus I} \left\| \triangle_{J'}^{\omega} \frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2 \\ &\quad + \sum_{k=1}^{\infty} \mathbb{P}^\alpha(\mathbf{1}_I\sigma) \left(c(A_k), |A_k|^{\frac{1}{n}} \right)^2 \sum_{\substack{F\in\mathcal{F}: \\ A_k\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \sum_{J'\in\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}:\ J'\subset I} \left\| \triangle_{J'}^{\omega} \frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2 \\ &\quad + \sum_{k=1}^{\infty} \mathbb{P}^\alpha(\mathbf{1}_I\sigma) \left(c(A_k), |A_k|^{\frac{1}{n}} \right)^2 \sum_{\substack{F\in\mathcal{F}: \\ A_k\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \sum_{J'\in\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}:\ I\subsetneqq J'\subset A_k} \left\| \triangle_{J'}^{\omega} \frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2 \\ &\equiv D_{\mathrm{disjoint}} + D_{\mathrm{descendent}} + D_{\mathrm{ancestor}} . \end{aligned}$$

We thus have from Lemma 15 again,

$$\begin{aligned} D_{\mathrm{disjoint}} &= \sum_{k=1}^{\infty} \mathbb{P}^\alpha(\mathbf{1}_I\sigma) \left(c(A_k), |A_k|^{\frac{1}{n}} \right)^2 \\ &\qquad \times \sum_{\substack{F\in\mathcal{F}: \\ A_k\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}} \sum_{J'\in\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}:\ J'\subset A_k\setminus I} \left\| \triangle_{J'}^{\omega} \frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2 \\ &\lesssim \sum_{k=1}^{\infty} \left(\frac{|I|_\sigma |A_k|^{\frac{1}{n}}}{|A_k|^{1+\frac{1-\alpha}{n}}} \right)^2 \boldsymbol{\tau}\ |A_k\setminus I|_\omega = \boldsymbol{\tau} \left\{ \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \sum_{k=1}^{\infty} \frac{|I|^{1-\frac{\alpha}{n}}}{|A_k|^{2\left(1-\frac{\alpha}{n}\right)}} |A_k\setminus I|_\omega \right\} |I|_\sigma \\ &\lesssim \boldsymbol{\tau} \left\{ \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \mathcal{P}^\alpha(I, \mathbf{1}_{I^c}\omega) \right\} |I|_\sigma \lesssim \boldsymbol{\tau} \mathcal{A}_2^{\alpha,*} |I|_\sigma , \end{aligned}$$

since

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{|I|^{1-\frac{\alpha}{n}}}{|A_k|^{2\left(1-\frac{\alpha}{n}\right)}}\left|A_k\setminus I\right|_{\omega} &= \int\sum_{k=1}^{\infty}\frac{|I|^{1-\frac{\alpha}{n}}}{|A_k|^{2\left(1-\frac{\alpha}{n}\right)}}\mathbf{1}_{A_k\setminus I}\left(x\right)d\omega\left(x\right)\\ &= \int\sum_{k=1}^{\infty}\frac{1}{2^{2\left(1-\frac{\alpha}{n}\right)k}}\frac{|I|^{1-\frac{\alpha}{n}}}{|I|^{2\left(1-\frac{\alpha}{n}\right)}}\mathbf{1}_{A_k\setminus I}\left(x\right)d\omega\left(x\right)\\ &\lesssim \int_{I^c}\left(\frac{|I|^{\frac{1}{n}}}{\left[|I|^{\frac{1}{n}}+\operatorname{quasidist}\left(x,I\right)\right]^2}\right)^{n-\alpha}d\omega\left(x\right)=\mathcal{P}^{\alpha}\left(I,\mathbf{1}_{I^c}\omega\right).\end{aligned}$$

The next term $D_{\mathrm{descendent}}$ satisfies

$$\begin{aligned}D_{\mathrm{descendent}} &\lesssim \sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}|A_k|^{\frac{1}{n}}}{|A_k|^{1+\frac{1-\alpha}{n}}}\right)^2\boldsymbol{\tau}\left\|\mathsf{P}_I^{\mathrm{good},\omega}\frac{\mathbf{x}}{2^k|I|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2\\ &= \boldsymbol{\tau}\sum_{k=1}^{\infty}2^{-2k(n-\alpha+1)}\left(\frac{|I|_{\sigma}}{|I|^{1-\frac{\alpha}{n}}}\right)^2\left\|\mathsf{P}_I^{\mathrm{good},\omega}\frac{\mathbf{x}}{|I|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2\\ &\lesssim \boldsymbol{\tau}\left\{\frac{|I|_{\sigma}\left\|\mathsf{P}_I^{\mathrm{good},\omega}\frac{\mathbf{x}}{|I|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2}{|I|^{2\left(1-\frac{\alpha}{n}\right)}}\right\}|I|_{\sigma}\lesssim\boldsymbol{\tau}A_2^{\alpha,\mathrm{energy}}|I|_{\sigma}\ .\end{aligned}$$

Finally for D_{ancestor} we note that each J' is of the form $J'=A_{\ell}\equiv\pi_{\Omega\mathcal{D}}^{(\ell)}I$ for some $\ell\geq 1$, and that there are at most $C\boldsymbol{\tau}$ pairs (F,A_k) with $k\geq\ell$ such that $A_k\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)$ and $J'=A_{\ell}\in\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}$. Now we write

$$\begin{aligned}D_{\mathrm{ancestor}} &= \sum_{k=1}^{\infty}\mathbb{P}^{\alpha}\left(\mathbf{1}_I\sigma\right)\left(c\left(A_k\right),|A_k|^{\frac{1}{n}}\right)^2\sum_{\substack{F\in\mathcal{F}:\\ A_k\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)}}\ \sum_{J'\in\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}-\mathrm{shift}}:\ I\subsetneqq J'\subset A_k}\left\|\triangle_{J'}^{\omega}\frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2\\ &\lesssim \boldsymbol{\tau}\sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}|A_k|^{\frac{1}{n}}}{|A_k|^{1+\frac{1-\alpha}{n}}}\right)^2\sum_{\ell=1}^{k}\left\|\triangle_{A_{\ell}}^{\omega}\frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2\\ &\leq \boldsymbol{\tau}\sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}|A_k|^{\frac{1}{n}}}{|A_k|^{1+\frac{1-\alpha}{n}}}\right)^2\left\|\mathsf{P}_{A_k}^{\mathrm{good},\omega}\frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2.\end{aligned}$$

At this point we need a *'prepare to puncture'* argument, as we will want to derive geometric decay from $\|\mathsf{P}_{J'}^{\omega}\mathbf{x}\|_{L^2(\omega)}^2$ by dominating it by the 'nonenergy' term $|J'|^{\frac{2}{n}}|J'\cap I|_{\omega}$, as well as using the Muckenhoupt energy constant. For this we define $\widetilde{\omega}=\omega-\omega(\{p\})\delta_p$ where p is an atomic point in I for which

$$\omega\left(\{p\}\right)=\sup_{q\in\mathfrak{P}_{(\sigma,\omega)}:\ q\in I}\omega\left(\{q\}\right).$$

(If ω has no atomic point in common with σ in I set $\widetilde{\omega}=\omega$.) Then we have $|I|_{\widetilde{\omega}}=\omega\left(I,\mathfrak{P}_{(\sigma,\omega)}\right)$ and

$$\frac{|I|_{\widetilde{\omega}}}{|I|^{\left(1-\frac{\alpha}{n}\right)}}\frac{|I|_{\sigma}}{|I|^{\left(1-\frac{\alpha}{n}\right)}}=\frac{\omega\left(I,\mathfrak{P}_{(\sigma,\omega)}\right)}{|I|^{\left(1-\frac{\alpha}{n}\right)}}\frac{|I|_{\sigma}}{|I|^{\left(1-\frac{\alpha}{n}\right)}}\leq A_2^{\alpha,\mathrm{punct}}.$$

A key observation, already noted in the proof of Lemma 3 above, is that

$$\|\triangle_K^{\omega}\mathbf{x}\|_{L^2(\omega)}^2=\left\{\begin{array}{ccc}\|\triangle_K^{\omega}(\mathbf{x}-\mathbf{p})\|_{L^2(\omega)}^2 & \text{if} & p\in K\\ \|\triangle_K^{\omega}\mathbf{x}\|_{L^2(\widetilde{\omega})}^2 & \text{if} & p\notin K\end{array}\right.\leq\ell\left(K\right)^2|K|_{\widetilde{\omega}},\qquad\text{for all }K\in\Omega\mathcal{D}\ ,\tag{9.17}$$

and so, as in the proof of Lemma 3,

$$\left\|\mathsf{P}_{A_k}^{\mathrm{good},\omega}\frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2\leq 3\,|A_k|_{\widetilde{\omega}}\ .$$

Then we continue with

$$\begin{aligned}
&\boldsymbol{\tau}\sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}\,|A_k|^{\frac{1}{n}}}{|A_k|^{1+\frac{1-\alpha}{n}}}\right)^2\left\|\mathsf{P}_{A_k}^{\mathrm{good},\omega}\frac{\mathbf{x}}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^2\\
\lesssim\;&\boldsymbol{\tau}\sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}\,|A_k|^{\frac{1}{n}}}{|A_k|^{1+\frac{1-\alpha}{n}}}\right)^2|A_k|_{\widetilde{\omega}}\\
=\;&\boldsymbol{\tau}\sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}}{|A_k|^{1-\frac{\alpha}{n}}}\right)^2|A_k\setminus I|_{\omega}+\boldsymbol{\tau}\sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}}{2^{k(n-\alpha)}\,|I|^{1-\frac{\alpha}{n}}}\right)^2|I|_{\widetilde{\omega}}\\
\lesssim\;&\boldsymbol{\tau}\left(\mathcal{A}_2^{\alpha,*}+A_2^{\alpha,\mathrm{punct}}\right)|I|_{\sigma}\,,
\end{aligned}$$

where the inequality $\sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}}{|A_k|^{1-\frac{\alpha}{n}}}\right)^2|A_k\setminus I|_{\omega}\lesssim\mathcal{A}_2^{\alpha,*}\,|I|_{\sigma}$ is already proved above in the estimate for D_{disjoint}.

9.3. **The backward Poisson testing inequality.** Fix $I\in\Omega\mathcal{D}$. It suffices to prove

$$\mathbf{Back}\left(\widehat{I}\right)\equiv\int_{\mathbb{R}^n}\left[\mathbb{Q}^{\alpha}\left(t\mathbf{1}_{\widehat{I}}\overline{\mu}\right)(y)\right]^2d\sigma(y)\lesssim\left\{\mathcal{A}_2^{\alpha}+\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}+\sqrt{A_2^{\alpha,\mathrm{energy}}}\right)\sqrt{A_2^{\alpha,\mathrm{punct}}}\right\}\int_{\widehat{I}}t^2d\overline{\mu}(x,t).\tag{9.18}$$

Note that in dimension $n=1$, Hytönen obtained in [Hyt2] the simpler bound A_2^{α} for the term analogous to (9.18). Here is a brief schematic diagram of the decompositions, with bounds in $\square$, used in this subsection:

$$\begin{array}{ccccc}
\mathbf{Back}\left(\widehat{I}\right) & & & & \\
\downarrow & & & & \\
U_s & & & & \\
\downarrow & & & & \\
T_s^{\mathrm{proximal}} & + & V_s^{\mathrm{remote}} & & \\
\boxed{\begin{array}{c}\mathcal{A}_2^{\alpha}+\\ \left(\mathcal{E}_{\alpha}^{\mathrm{plug}}+\sqrt{A_2^{\alpha,\mathrm{energy}}}\right)\sqrt{A_2^{\alpha,\mathrm{punct}}}\end{array}} & & \downarrow & & \\
 & & \downarrow & & \\
 & & T_s^{\mathrm{difference}} & + & T_s^{\mathrm{intersection}} \\
 & & \boxed{\begin{array}{c}\mathcal{A}_2^{\alpha}+\\ \left(\mathcal{E}_{\alpha}^{\mathrm{plug}}+\sqrt{A_2^{\alpha,\mathrm{energy}}}\right)\sqrt{A_2^{\alpha,\mathrm{punct}}}\end{array}} & & \boxed{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}+\sqrt{A_2^{\alpha,\mathrm{energy}}}\right)\sqrt{A_2^{\alpha,\mathrm{punct}}}}
\end{array}.$$

Using (9.10) we see that the integral on the right hand side of (9.18) is

$$\int_{\widehat{I}}t^2d\overline{\mu}=\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F):\ J\subset I}\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\,.\tag{9.19}$$

where $\mathsf{P}_{F,J}^{\omega}$ was defined earlier.

We now compute using (9.10) again that

$$\begin{aligned}
\mathbb{Q}^{\alpha}\left(t\mathbf{1}_{\widehat{I}}\overline{\mu}\right)(y)&=\int_{\widehat{I}}\frac{t^2}{\left(t^2+|x-y|^2\right)^{\frac{n+1-\alpha}{2}}}d\overline{\mu}(x,t)\\
&\approx\sum_{F\in\mathcal{F}}\sum_{\substack{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)\\ J\subset I}}\frac{\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(|J|^{\frac{1}{n}}+|y-c_J|\right)^{n+1-\alpha}},
\end{aligned}\tag{9.20}$$

and then expand the square and integrate to obtain that the term $\mathbf{Back}\left(\widehat{I}\right)$ is

$$\sum_{\substack{F\in\mathcal{F}\\ J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F)\\ J\subset I}}\sum_{\substack{F'\in\mathcal{F}:\\ J'\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(F'\right)\\ J'\subset I}}\int_{\mathbb{R}^n}\frac{\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{n+1-\alpha}}\frac{\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(\left|J'\right|^{\frac{1}{n}}+\left|y-c_{J'}\right|\right)^{n+1-\alpha}}d\sigma\left(y\right).$$

By symmetry we may assume that $\ell\left(J'\right)\leq\ell\left(J\right)$. We fix a nonnegative integer s, and consider those quasicubes J and J' with $\ell\left(J'\right)=2^{-s}\ell\left(J\right)$. For fixed s we will control the expression

$$\begin{aligned}U_s &\equiv \sum_{F,F'\in\mathcal{F}}\sum_{\substack{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F),\ J'\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(F'\right)\\ J,J'\subset I,\ \ell\left(J'\right)=2^{-s}\ell(J)}}\\ &\quad\times\int_{\mathbb{R}^n}\frac{\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{n+1-\alpha}}\frac{\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(\left|J'\right|^{\frac{1}{n}}+\left|y-c_{J'}\right|\right)^{n+1-\alpha}}d\sigma\left(y\right),\end{aligned}$$

by proving that

$$U_s\lesssim 2^{-\delta s}\left\{\mathcal{A}_2^{\alpha}+\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}+\sqrt{A_2^{\alpha,\mathrm{energy}}}\right)\sqrt{A_2^{\alpha,\mathrm{punct}}}\right\}\int_{\widehat{I}}t^2d\overline{\mu},\qquad\text{where }\delta=\frac{1}{2n}.\tag{9.21}$$

With this accomplished, we can sum in $s\geq 0$ to control the term $\mathbf{Back}\left(\widehat{I}\right)$. We now decompose $U_s=T_s^{\mathrm{proximal}}+T_s^{\mathrm{difference}}+T_s^{\mathrm{intersection}}$ into three pieces.

Our first decomposition is to write

$$U_s=T_s^{\mathrm{proximal}}+V_s^{\mathrm{remote}}\ ,\tag{9.22}$$

where in the 'proximal' term T_s^{proximal} we restrict the summation over pairs of quasicubes J,J' to those satisfying $\mathrm{qdist}\left(c_J,c_{J'}\right)<2^{s\delta}\ell\left(J\right)$; while in the 'remote' term V_s^{remote} we restrict the summation over pairs of quasicubes J,J' to those satisfying the opposite inequality $\mathrm{qdist}\left(c_J,c_{J'}\right)\geq 2^{s\delta}\ell\left(J\right)$. Then we further decompose

$$V_s^{\mathrm{remote}}=T_s^{\mathrm{difference}}+T_s^{\mathrm{intersection}},$$

where in the 'difference' term $T_s^{\mathrm{difference}}$ we restict integration in y to the difference $\mathbb{R}^n\setminus B\left(J,J'\right)$ of $\mathbb{R}^n$ and

$$B\left(J,J'\right)\equiv B\left(c_J,\frac{1}{2}\,\mathrm{qdist}\left(c_J,c_{J'}\right)\right),$$

the quasiball centered at c_J with radius $\frac{1}{2}\,\mathrm{qdist}\left(c_J,c_{J'}\right)$; while in the 'intersection' term $T_s^{\mathrm{intersection}}$ we restict integration in y to the intersection $\mathbb{R}^n\cap B\left(J,J'\right)$ of $\mathbb{R}^n$ with the quasiball $B\left(J,J'\right)$; i.e.

$$\begin{aligned}T_s^{\mathrm{intersection}} &\equiv \sum_{F,F'\in\mathcal{F}}\sum_{\substack{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F),\ J'\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(F'\right)\\ J,J'\subset I,\ \ell\left(J'\right)=2^{-s}\ell(J)\\ \mathrm{qdist}(c_J,c_{J'})\geq 2^{s(1+\delta)}\ell\left(J'\right)}}\\ &\quad\times\int_{B(J,J')}\frac{\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{n+1-\alpha}}\frac{\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(\left|J'\right|^{\frac{1}{n}}+\left|y-c_{J'}\right|\right)^{n+1-\alpha}}d\sigma\left(y\right).\end{aligned}$$

Here is a schematic reminder of the these decompositions with the distinguishing points of the definitions boxed:

$$\boxed{\begin{array}{ccccc} U_s & & & & \\ \downarrow & & & & \\ T_s^{\text{proximal}} & + & V_s^{\text{remote}} & & \\ \boxed{\operatorname{qdist}(c_J, c_{J'}) < 2^{s\delta}\ell(J)} & & \boxed{\operatorname{qdist}(c_J, c_{J'}) \geq 2^{s\delta}\ell(J)} & & \\ & & \downarrow & & \\ & & T_s^{\text{difference}} & + & T_s^{\text{intersection}} \\ & & \boxed{\int_{\mathbb{R}^n \setminus B(J,J')}} & & \boxed{\boxed{\int_{B(J,J')}}} \end{array}}.$$

We will exploit the restriction of integration to $B(J, J')$, together with the condition

$$\operatorname{qdist}(c_J, c_{J'}) \geq 2^{s(1+\delta)}\ell(J') = 2^{s\delta}\ell(J),$$

in establishing (**??**) below, which will then give an estimate for the term $T_s^{\text{intersection}}$ using an argument dual to that used for the other terms T_s^{proximal} and $T_s^{\text{difference}}$, to which we now turn.

9.3.1. *The proximal and difference terms.* We have

$$\begin{aligned} T_s^{\text{proximal}} &\equiv \sum_{F,F' \in \mathcal{F}} \sum_{\substack{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F),\ J' \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F') \\ J,J' \subset I,\ \ell(J') = 2^{-s}\ell(J) \text{ and } \operatorname{qdist}(c_J, c_{J'}) < 2^{s\delta}\ell(J)}} \\ &\quad \times \int_{\mathbb{R}^n} \frac{\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(|J|^{\frac{1}{n}} + |y - c_J|\right)^{n+1-\alpha}} \frac{\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(|J'|^{\frac{1}{n}} + |y - c_{J'}|\right)^{n+1-\alpha}} d\sigma(y) \\ &\leq M_s^{\text{proximal}} \sum_{F \in \mathcal{F}} \sum_{\substack{\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F) \\ J \subset I}} \left\|\mathsf{P}_{F,J}^{\omega}\mathbf{z}\right\|_{\omega}^2 = M_s^{\text{proximal}} \int_{\widehat{I}} t^2 d\overline{\mu}, \end{aligned}$$

where

$$\begin{aligned} M_s^{\text{proximal}} &\equiv \sup_{F \in \mathcal{F}} \sup_{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F)} A_s^{\text{proximal}}(J); \\ A_s^{\text{proximal}}(J) &\equiv \sum_{F' \in \mathcal{F}} \sum_{\substack{J' \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F') \\ J' \subset I,\ \ell(J') = 2^{-s}\ell(J) \text{ and } \operatorname{qdist}(c_J, c_{J'}) < 2^{s\delta}\ell(J)}} \int_{\mathbb{R}^n} S_{(J',J)}^{F'}(y)\, d\sigma(y); \\ S_{(J',J)}^{F'}(x) &\equiv \frac{1}{\left(|J|^{\frac{1}{n}} + |y - c_J|\right)^{n+1-\alpha}} \frac{\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(|J'|^{\frac{1}{n}} + |y - c_{J'}|\right)^{n+1-\alpha}}, \end{aligned}$$

and similarly

$$\begin{aligned} T_s^{\text{difference}} &\equiv \sum_{F,F' \in \mathcal{F}} \sum_{\substack{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F),\ J' \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F') \\ J,J' \subset I,\ \ell(J') = 2^{-s}\ell(J) \text{ and } \operatorname{qdist}(c_J, c_{J'}) \geq 2^{s\delta}\ell(J)}} \\ &\quad \times \int_{\mathbb{R}^n \setminus B(J,J')} \frac{\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(|J|^{\frac{1}{n}} + |y - c_J|\right)^{n+1-\alpha}} \frac{\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(|J'|^{\frac{1}{n}} + |y - c_{J'}|\right)^{n+1-\alpha}} d\sigma(y) \\ &\leq M_s^{\text{difference}} \sum_{F \in \mathcal{F}} \sum_{\substack{\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F) \\ J \subset I}} \left\|\mathsf{P}_{F,J}^{\omega}\mathbf{z}\right\|_{\omega}^2 = M_s^{\text{difference}} \int_{\widehat{I}} t^2 d\overline{\mu}; \end{aligned}$$

where

$$\begin{aligned} M_s^{\text{difference}} &\equiv \sup_{F\in\mathcal{F}} \sup_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F)} A_s^{\text{difference}}(J) ; \\ A_s^{\text{difference}}(J) &\equiv \sum_{F'\in\mathcal{F}} \sum_{\substack{J'\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F') \\ J'\subset I,\ \ell(J')=2^{-s}\ell(J) \text{ and } \operatorname{qdist}(c_J,c_{J'})\geq 2^{s\delta}\ell(J)}} \int_{\mathbb{R}^n\setminus B(J,J')} S_{(J',J)}^{F'}(y)\,d\sigma(y) . \end{aligned}$$

The restriction of integration in $A_s^{\text{difference}}$ to $\mathbb{R}^n\setminus B(J,J')$ will be used to establish (9.24) below.

Notation 3. *Since the quasicubes F,J,F',J' that arise in all of the sums here satisfy (recall (9.10))*

$$J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F),\ \ J'\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F') \ \text{ and } \ell(J')=2^{-s}\ell(J) \ \text{ and } J,J'\subset I,$$

we will often employ the notation $\overset{*}{\sum}$ to remind the reader that, as applicable, these four conditions are in force even when they are not explictly mentioned.

Now fix J as in M_s^{proximal} respectively $M_s^{\text{difference}}$, and decompose the sum over J' in $A_s^{\text{proximal}}(J)$ respectively $A_s^{\text{difference}}(J)$ by

$$\begin{aligned} A_s^{\text{proximal}}(J) &= \sum_{F'\in\mathcal{F}} \sum_{\substack{J'\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F') \\ J'\subset I,\ \ell(J')=2^{-s}\ell(J) \text{ and } \operatorname{qdist}(c_J,c_{J'})<2^{s\delta}\ell(J)}} \int_{\mathbb{R}^n} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\ &= \sum_{F'\in\mathcal{F}} \overset{*}{\sum_{\substack{c_{J'}\in 2J \\ \operatorname{qdist}(c_J,c_{J'})<2^{s\delta}\ell(J)}}} \int_{\mathbb{R}^n} S_{(J',J)}^{F'}(y)\,d\sigma(y) + \sum_{F'\in\mathcal{F}}\sum_{\ell=1}^{\infty} \overset{*}{\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J \\ \operatorname{qdist}(c_J,c_{J'})<2^{s\delta}\ell(J)}}} \int_{\mathbb{R}^n} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\ &\equiv \sum_{\ell=0}^{\infty} A_s^{\text{proximal},\ell}(J) , \end{aligned}$$

respectively

$$\begin{aligned} A_s^{\text{difference}}(J) &= \sum_{F'\in\mathcal{F}} \sum_{\substack{J'\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(F') \\ J'\subset I,\ \ell(J')=2^{-s}\ell(J) \text{ and } \operatorname{qdist}(c_J,c_{J'})\geq 2^{s\delta}\ell(J)}} \int_{\mathbb{R}^n\setminus B(J,J')} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\ &= \sum_{F'\in\mathcal{F}} \overset{*}{\sum_{\substack{c_{J'}\in 2J \\ \operatorname{qdist}(c_J,c_{J'})\geq 2^{s\delta}\ell(J)}}} \int_{\mathbb{R}^n\setminus B(J,J')} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\ &\quad + \sum_{\ell=1}^{\infty}\sum_{F'\in\mathcal{F}} \overset{*}{\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J \\ \operatorname{qdist}(c_J,c_{J'})\geq 2^{s\delta}\ell(J)}}} \int_{\mathbb{R}^n\setminus B(J,J')} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\ &\equiv \sum_{\ell=0}^{\infty} A_s^{\text{difference},\ell}(J) . \end{aligned}$$

Let m be the smallest integer for which

$$2^{-m}\sqrt{n}\leq\frac{1}{3}. \tag{9.23}$$

Now decompose the integrals over I in $A_s^{\text{proximal},\ell}(J)$ by

$$\begin{aligned}
A_s^{\text{proximal},0}(J) &= \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2J \\ \operatorname{qdist}(c_J,c_{J'})<2^{s\delta}\ell(J)}} \int_{\mathbb{R}^n\setminus 4J} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\quad + \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2J \\ \operatorname{qdist}(c_J,c_{J'})<2^{s\delta}\ell(J)}} \int_{4J} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\equiv A_{s,far}^{\text{proximal},0}(J) + A_{s,near}^{\text{proximal},0}(J), \\
A_s^{\text{proximal},\ell}(J) &= \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J \\ \operatorname{qdist}(c_J,c_{J'})<2^{s\delta}\ell(J)}} \int_{\mathbb{R}^n\setminus 2^{\ell+2}J} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\quad + \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J \\ \operatorname{qdist}(c_J,c_{J'})<2^{s\delta}\ell(J)}} \int_{2^{\ell+2}J\setminus 2^{\ell-m}J} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\quad + \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J \\ \operatorname{qdist}(c_J,c_{J'})<2^{s\delta}\ell(J)}} \int_{2^{\ell-m}J} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\equiv A_{s,far}^{\text{proximal},\ell}(J) + A_{s,near}^{\text{proximal},\ell}(J) + A_{s,close}^{\text{proximal},\ell}(J), \qquad \ell\geq 1.
\end{aligned}$$

Similarly we decompose the integrals over the difference

$$I^*\equiv\mathbb{R}^n\setminus B(J,J')$$

in $A_s^{\text{difference},\ell}(J)$ by

$$\begin{aligned}
A_s^{\text{difference},0}(J) &= \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2J \\ \operatorname{qdist}(c_J,c_{J'})\geq 2^{s\delta}\ell(J)}} \int_{I^*\setminus 4J} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\quad + \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2J \\ \operatorname{qdist}(c_J,c_{J'})\geq 2^{s\delta}\ell(J)}} \int_{I^*\cap 4J} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\equiv A_{s,far}^{\text{difference},0}(J) + A_{s,near}^{\text{difference},0}(J), \\
A_s^{\text{difference},\ell}(J) &= \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J \\ \operatorname{qdist}(c_J,c_{J'})\geq 2^{s\delta}\ell(J)}} \int_{I^*\setminus 2^{\ell+2}J} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\quad + \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J \\ \operatorname{qdist}(c_J,c_{J'})\geq 2^{s\delta}\ell(J)}} \int_{I^*\cap(2^{\ell+2}J\setminus 2^{\ell-m}J)} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\quad + \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J \\ \operatorname{qdist}(c_J,c_{J'})\geq 2^{s\delta}\ell(J)}} \int_{I^*\cap 2^{\ell-m}J} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\equiv A_{s,far}^{\text{difference},\ell}(J) + A_{s,near}^{\text{difference},\ell}(J) + A_{s,close}^{\text{difference},\ell}(J), \qquad \ell\geq 1.
\end{aligned}$$

We now note the important point that the close terms $A_{s,close}^{\text{proximal},\ell}(J)$ and $A_{s,close}^{\text{difference},\ell}(J)$ both *vanish* for $\ell>\delta s$ because of the decomposition (9.22):

$$A_{s,close}^{\text{proximal},\ell}(J) = A_{s,close}^{\text{difference},\ell}(J) = 0, \qquad \ell\geq 1+\delta s. \tag{9.24}$$

Indeed, if $c_{J'} \in 2^{\ell+1}J \setminus 2^{\ell}J$, then we have

$$\frac{1}{2}2^{\ell}\ell(J) \leq \operatorname{qdist}(c_J, c_{J'}), \tag{9.25}$$

and if $\ell \geq 1+\delta s$, then

$$\operatorname{qdist}(c_J, c_{J'}) \geq 2^{\delta s}\ell(J) = 2^{(1+\delta)s}\ell(J').$$

It now follows from the definition of V_s^{remote} and T_s^{proximal} in (9.22), that $A_{s,close}^{\text{proximal},\ell}(J)=0$, and so we are left to consider the term $A_{s,close}^{\text{difference},\ell}(J)$, where the integration is taken over the set $\mathbb{R}^n \setminus B(J,J')$. But we are also restricted in $A_{s,close}^{\text{difference},\ell}(J)$ to integrating over the quasicube $2^{\ell-m}J$, which is contained in $B(J,J')$ by (9.25). Indeed, the smallest ball centered at c_J that contains $2^{\ell-m}J$ has radius $\sqrt{n}\frac{1}{2}2^{\ell-m}\ell(J)$, which by (9.23) and (9.25) is at most $\frac{1}{4}2^{\ell}\ell(J) \leq \frac{1}{2}\operatorname{qdist}(c_J,c_{J'})$, the radius of $B(J,J')$. Thus the range of integration in the term $A_{s,close}^{\text{difference},\ell}(J)$ is the empty set, and so $A_{s,close}^{\text{difference},\ell}(J)=0$ as well as $A_{s,close}^{\text{proximal},\ell}(J)=0$. This proves (9.24).

From now on we treat T_s^{proximal} and $T_s^{\text{difference}}$ in the same way since the terms $A_{s,close}^{\text{proximal},\ell}(J)$ and $A_{s,close}^{\text{difference},\ell}(J)$ both vanish for $\ell \geq 1+\delta s$. Thus we will suppress the superscripts proximal and difference in the *far*, *near* and *close* decomposition of $A_s^{\text{proximal},\ell}(J)$ and $A_s^{\text{difference},\ell}(J)$, and we will also suppress the conditions $\operatorname{qdist}(c_J,c_{J'}) < 2^{s\delta}\ell(J)$ and $\operatorname{qdist}(c_J,c_{J'}) \geq 2^{s\delta}\ell(J)$ in the proximal and difference terms since they no longer play a role. Using the bounded overlap of the shifted coronas $\mathcal{C}_F^{\text{good},\boldsymbol{\tau}-\text{shift}}$, we have

$$\sum_{F'\in\mathcal{F}} \left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \lesssim \boldsymbol{\tau}\left|J'\right|^{\frac{2}{n}}\left|J'\right|_{\omega} ,$$

and so we have

$$\begin{aligned}
A_{s,far}^{0}(J) &\leq \sum_{F'\in\mathcal{F}}\sum_{c_{J'}\in 2J}^{*}\int_{\mathbb{R}^n\setminus 4J} S_{(J',J)}^{F'}(y)\,d\sigma(y)\\
&\lesssim \boldsymbol{\tau}\sum_{c_{J'}\in 2J}\int_{\mathbb{R}^n\setminus 4J}\frac{\left|J'\right|^{\frac{2}{n}}\left|J'\right|_{\omega}}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{2(n+1-\alpha)}}d\sigma(y)\\
&= \boldsymbol{\tau}2^{-2s}\left(\sum_{c_{J'}\in 2J}\left|J'\right|_{\omega}\right)\int_{\mathbb{R}^n\setminus 4J}\frac{\left|J\right|^{\frac{2}{n}}}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{2(n+1-\alpha)}}d\sigma(y),
\end{aligned}$$

which is dominated by

$$\begin{aligned}
&\boldsymbol{\tau}2^{-2s}\left|4J\right|_{\omega}\int_{\mathbb{R}^n\setminus 4J}\frac{1}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{2(n-\alpha)}}d\sigma(y)\\
\approx\ &\boldsymbol{\tau}2^{-2s}\frac{\left|4J\right|_{\omega}}{\left|4J\right|^{1-\frac{\alpha}{n}}}\int_{\mathbb{R}^n\setminus 4J}\left(\frac{\left|J\right|^{\frac{1}{n}}}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{2}}\right)^{n-\alpha}d\sigma(y)\\
\lesssim\ &\boldsymbol{\tau}2^{-2s}\frac{\left|4J\right|_{\omega}}{\left|4J\right|^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}\left(4J,\mathbf{1}_{\mathbb{R}^n\setminus 4J}\sigma\right) \lesssim \boldsymbol{\tau}2^{-2s}\mathcal{A}_2^{\alpha} .
\end{aligned}$$

To estimate the near term $A_{s,near}^{0}(J)$, we initially keep the energy $\left\Vert \mathsf{P}_{F',J'}^{\omega}\mathbf{z}\right\Vert_{L^{2}(\omega)}^{2}$ and write

$$
\begin{aligned}
(9.26)\quad A_{s,near}^{0}(J) &\leq \sum_{F'\in\mathcal{F}}\sum_{c_{J'}\in 2J}^{*}\int_{4J}S_{(J',J)}^{F'}(y)\,d\sigma(y)\\
&\approx \sum_{F'\in\mathcal{F}}\sum_{c_{J'}\in 2J}^{*}\int_{4J}\frac{1}{|J|^{\frac{1}{n}(n+1-\alpha)}}\frac{\left\Vert \mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\Vert_{L^{2}(\omega)}^{2}}{\left(|J'|^{\frac{1}{n}}+|y-c_{J'}|\right)^{n+1-\alpha}}d\sigma(y)\\
&= \sum_{F'\in\mathcal{F}}\frac{1}{|J|^{\frac{1}{n}(n+1-\alpha)}}\sum_{c_{J'}\in 2J}^{*}\left\Vert \mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\Vert_{L^{2}(\omega)}^{2}\int_{4J}\frac{1}{\left(|J'|^{\frac{1}{n}}+|y-c_{J'}|\right)^{n+1-\alpha}}d\sigma(y)\\
&= \sum_{F'\in\mathcal{F}}\frac{1}{|J|^{\frac{1}{n}(n+1-\alpha)}}\sum_{c_{J'}\in 2J}^{*}\left\Vert \mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\Vert_{L^{2}(\omega)}^{2}\frac{\mathrm{P}^{\alpha}\left(J',\mathbf{1}_{I\cap(4J)}\sigma\right)}{|J'|^{\frac{1}{n}}}.
\end{aligned}
$$

In order to estimate the final sum above, we must invoke the 'prepare to puncture' argument above, as we will want to derive geometric decay from $\Vert\mathsf{P}_{J'}^{\omega}\mathbf{x}\Vert_{L^{2}(\omega)}^{2}$ by dominating it by the 'nonenergy' term $|J'|^{\frac{2}{n}}|J'|_{\omega}$, as well as using the Muckenhoupt energy constant. Choose an alternate quasicube $\widetilde{J}\in\mathcal{A}\Omega\mathcal{D}$ satisfying $\bigcup_{c_{J'}\in 2J}J'\subset 4J\subset\widetilde{J}$ and $\ell\left(\widetilde{J}\right)\leq C\ell(J)$. Define $\widetilde{\omega}=\omega-\omega(\{p\})\,\delta_{p}$ where p is an atomic point in $\widetilde{J}$ for which

$$
\omega(\{p\})=\sup_{q\in\mathfrak{P}_{(\sigma,\omega)}:\ q\in\widetilde{J}}\omega(\{q\}).
$$

(If ω has no atomic point in common with σ in $\widetilde{J}$ set $\widetilde{\omega}=\omega$.) Then we have $\left|\widetilde{J}\right|_{\widetilde{\omega}}=\omega\left(\widetilde{J},\mathfrak{P}_{(\sigma,\omega)}\right)$ and

$$
\frac{\left|\widetilde{J}\right|_{\widetilde{\omega}}}{\left|\widetilde{J}\right|^{\left(1-\frac{\alpha}{n}\right)}}\frac{\left|\widetilde{J}\right|_{\sigma}}{\left|\widetilde{J}\right|^{\left(1-\frac{\alpha}{n}\right)}}=\frac{\omega\left(\widetilde{J},\mathfrak{P}_{(\sigma,\omega)}\right)}{\left|\widetilde{J}\right|^{\left(1-\frac{\alpha}{n}\right)}}\frac{\left|\widetilde{J}\right|_{\sigma}}{\left|\widetilde{J}\right|^{\left(1-\frac{\alpha}{n}\right)}}\leq A_{2}^{\alpha,\mathrm{punct}}.
$$

From (9.17) we also have

$$
\sum_{F'\in\mathcal{F}}\left\Vert \mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\Vert_{L^{2}(\omega)}^{2}\lesssim\boldsymbol{\tau}\ell(J')^{2}|J'|_{\widetilde{\omega}}\ ,
$$

for all J' arising in the sum in the final line of (9.26) above.

Now by Cauchy-Schwarz and the alternate local estimate (9.16) in Lemma 18 with $M=\widetilde{J}$ applied to the second line below, the last sum in (9.26) is dominated by

$$
\begin{aligned}
(9.27)\quad &\frac{1}{|J|^{\frac{1}{n}(n+1-\alpha)}}\left(\sum_{F'\in\mathcal{F}}\sum_{c(J')\in 2J}^{*}\left\Vert \mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\Vert_{L^{2}(\omega)}^{2}\right)^{\frac{1}{2}}\\
&\quad\times\left(\sum_{F'\in\mathcal{F}}\sum_{c_{J'}\in 2J}^{*}\left\Vert \mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\Vert_{L^{2}(\omega)}^{2}\left(\frac{\mathrm{P}^{\alpha}\left(J',\mathbf{1}_{I\cap(4J)}\sigma\right)}{|J'|^{\frac{1}{n}}}\right)^{2}\right)^{\frac{1}{2}}\\
&\lesssim \frac{1}{|J|^{\frac{1}{n}(n+1-\alpha)}}\left(\boldsymbol{\tau}\sum_{c_{J'}\in 2J}|J'|^{\frac{2}{n}}|J'|_{\widetilde{\omega}}\right)^{\frac{1}{2}}\sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^{2}+A_{2}^{\alpha,\mathrm{energy}}}\sqrt{\boldsymbol{\tau}\left|\widetilde{J}\right|_{\sigma}}\\
&\lesssim \boldsymbol{\tau}\frac{2^{-s}|J|^{\frac{1}{n}}}{|J|^{\frac{1}{n}(n+1-\alpha)}}\sqrt{|4J|_{\widetilde{\omega}}}\sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^{2}+A_{2}^{\alpha,\mathrm{energy}}}\sqrt{\left|\widetilde{J}\right|_{\sigma}}\\
&\lesssim \boldsymbol{\tau}2^{-s}\sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^{2}+A_{2}^{\alpha,\mathrm{energy}}}\sqrt{\frac{\left|\widetilde{J}\right|_{\widetilde{\omega}}}{\left|\widetilde{J}\right|^{\frac{1}{n}(n-\alpha)}}\frac{\left|\widetilde{J}\right|_{\sigma}}{\left|\widetilde{J}\right|^{\frac{1}{n}(n-\alpha)}}}\lesssim\boldsymbol{\tau}2^{-s}\sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^{2}+A_{2}^{\alpha,\mathrm{energy}}}\sqrt{A_{2}^{\alpha,\mathrm{punct}}}\ .
\end{aligned}
$$

Similarly, for $\ell \geq 1$, we can estimate the far term

$$\begin{aligned}
A^{\ell}_{s,far}\left(J\right) &\leq \sum_{F'\in\mathcal{F}} \sum^{*}_{c_{J'}\in\left(2^{\ell+1}J\right)\setminus\left(2^{\ell}J\right)} \int_{\mathbb{R}^n\setminus 2^{\ell+2}J} S^{F'}_{\left(J',J\right)}\left(y\right)d\sigma\left(y\right) \\
&\lesssim \boldsymbol{\tau} \sum_{c_{J'}\in\left(2^{\ell+1}J\right)\setminus\left(2^{\ell}J\right)} \int_{\mathbb{R}^n\setminus 2^{\ell+2}J} \frac{\left|J'\right|^{\frac{2}{n}}\left|J'\right|_{\omega}}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{2(n+1-\alpha)}}d\sigma\left(y\right) \\
&= \boldsymbol{\tau} 2^{-2s}\left(\sum_{c_{J'}\in\left(2^{\ell+1}J\right)}\left|J'\right|_{\omega}\right)\int_{\mathbb{R}^n\setminus 2^{\ell+2}J} \frac{\left|J\right|^{\frac{2}{n}}}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{2(n+1-\alpha)}}d\sigma\left(y\right) \\
&\approx \boldsymbol{\tau} 2^{-2s}2^{-2\ell}\left(\sum_{c_{J'}\in\left(2^{\ell+1}J\right)}\left|J'\right|_{\omega}\right)\int_{\mathbb{R}^n\setminus 2^{\ell+2}J} \frac{\left|2^{\ell}J\right|^{\frac{2}{n}}}{\left(\left|2^{\ell}J\right|^{\frac{1}{n}}+\left|y-c_{2^{\ell}J}\right|\right)^{2(n+1-\alpha)}}d\sigma\left(y\right),
\end{aligned}$$

which is at most

$$\begin{aligned}
& \boldsymbol{\tau} 2^{-2s}2^{-2\ell}\left|2^{\ell+2}J\right|_{\omega}\int_{\mathbb{R}^n\setminus 2^{\ell+2}J} \frac{1}{\left(\left|2^{\ell}J\right|^{\frac{1}{n}}+\left|y-c_{2^{\ell}J}\right|\right)^{2(n-\alpha)}}d\sigma\left(y\right) \\
\approx\ & \boldsymbol{\tau} 2^{-2s}2^{-2\ell}\frac{\left|2^{\ell+2}J\right|_{\omega}}{\left|2^{\ell}J\right|^{1-\frac{\alpha}{n}}}\int_{\mathbb{R}^n\setminus 2^{\ell+2}J}\left(\frac{\left|2^{\ell}J\right|^{\frac{1}{n}}}{\left(\left|2^{\ell}J\right|^{\frac{1}{n}}+\left|y-c_{2^{\ell}J}\right|\right)^{2}}\right)^{n-\alpha}d\sigma\left(y\right) \\
\lesssim\ & \boldsymbol{\tau} 2^{-2s}2^{-2\ell}\left\{\frac{\left|2^{\ell+2}J\right|_{\omega}}{\left|2^{\ell}J\right|^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}\left(2^{\ell+2}J,\mathbf{1}_{\mathbb{R}^n\setminus 2^{\ell+2}J}\sigma\right)\right\}\lesssim \boldsymbol{\tau} 2^{-2s}2^{-2\ell}\mathcal{A}^{\alpha}_{2}\ .
\end{aligned}$$

To estimate the near term $A^{\ell}_{s,near}\left(J\right)$ we must again invoke the *'prepare to puncture'* argument. Choose an alternate quasicube $\widetilde{J}\in\mathcal{A}\Omega\mathcal{D}$ such that $\bigcup_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J} J'\subset 2^{\ell+2}J\subset\widetilde{J}$ and $\ell\left(\widetilde{J}\right)\leq C2^{\ell}\ell\left(J\right)$. Define $\widetilde{\omega}=\omega-\omega\left(\{p\}\right)\delta_p$ where p is an atomic point in $\widetilde{J}$ for which

$$\omega\left(\{p\}\right)=\sup_{q\in\mathfrak{P}_{(\sigma,\omega)}:\ q\in\widetilde{J}}\omega\left(\{q\}\right).$$

(If ω has no atomic point in common with σ in $\widetilde{J}$ set $\widetilde{\omega}=\omega$.) Then we have $\left|\widetilde{J}\right|_{\widetilde{\omega}}=\omega\left(\widetilde{J},\mathfrak{P}_{(\sigma,\omega)}\right)$, and just as in the argument above following (9.26), we have from (9.17) both

$$\frac{\left|\widetilde{J}\right|_{\widetilde{\omega}}}{\left|\widetilde{J}\right|^{\left(1-\frac{\alpha}{n}\right)}}\frac{\left|\widetilde{J}\right|_{\sigma}}{\left|\widetilde{J}\right|^{\left(1-\frac{\alpha}{n}\right)}}\leq A^{\alpha,\text{punct}}_{2} \text{ and } \sum_{F'\in\mathcal{F}}\left\|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\right\|^{2}_{L^2(\omega)}\lesssim\boldsymbol{\tau}\ell\left(J'\right)^{2}\left|J'\right|_{\widetilde{\omega}}\ .$$

Thus using that $m=\left[\log_2\left(3\sqrt{n}\right)\right]+1$ in the definition of $A^{\ell}_{s,near}\left(J\right)$, we see that

$$\begin{aligned}
A_{s,near}^{\ell}(J) &\leq \sum_{F'\in\mathcal{F}} \sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*} \int_{2^{\ell+2}J\setminus 2^{\ell-m}J} S_{(J',J)}^{F'}(y)\, d\sigma(y) \\
&\approx \sum_{F'\in\mathcal{F}} \sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*} \int_{2^{\ell+2}J\setminus 2^{\ell-m}J} \frac{1}{\left|2^{\ell}J\right|^{\frac{1}{n}(n+1-\alpha)}} \frac{\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(\left|J'\right|^{\frac{1}{n}}+\left|y-c_{J'}\right|\right)^{n+1-\alpha}} d\sigma(y) \\
&\lesssim \frac{1}{\left|2^{\ell}J\right|^{\frac{1}{n}(n+1-\alpha)}} \sum_{F'\in\mathcal{F}} \sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*} \left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\
&\qquad\times \int_{2^{\ell+2}J} \frac{1}{\left(\left|J'\right|^{\frac{1}{n}}+\left|y-c_{J'}\right|\right)^{n+1-\alpha}} d\sigma(y),
\end{aligned}$$

is dominated by

$$\begin{aligned}
&\frac{1}{\left|2^{\ell}J\right|^{\frac{1}{n}(n+1-\alpha)}} \sum_{F'\in\mathcal{F}} \sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*} \left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \frac{\mathrm{P}^{\alpha}\left(J',\mathbf{1}_{2^{\ell+2}J}\sigma\right)}{\left|J'\right|^{\frac{1}{n}}} \\
\leq\ & \frac{1}{\left|2^{\ell}J\right|^{\frac{1}{n}(n+1-\alpha)}} \left(\sum_{F'\in\mathcal{F}} \sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*} \left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\right)^{\frac{1}{2}} \\
&\times \left(\sum_{F'\in\mathcal{F}} \sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*} \left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \left(\frac{\mathrm{P}^{\alpha}\left(J',\mathbf{1}_{2^{\ell+2}J}\sigma\right)}{\left|J'\right|^{\frac{1}{n}}}\right)^2\right)^{\frac{1}{2}}.
\end{aligned}$$

This can now be estimated using $\sum_{F'\in\mathcal{F}} \left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \lesssim \boldsymbol{\tau}\left|J'\right|^{\frac{2}{n}}\left|J'\right|_{\widetilde{\omega}} = \boldsymbol{\tau}2^{-2s}\left|J\right|^{\frac{2}{n}}\left|J'\right|_{\widetilde{\omega}}$ along with the alternate local estimate (9.16) in Lemma 18 with $M=\widetilde{J}$ applied to the final line above to get

$$\begin{aligned}
A_{s,near}^{\ell}(J) &\lesssim \boldsymbol{\tau}2^{-s}2^{-\ell}\frac{\left|2^{\ell}J\right|^{\frac{1}{n}}}{\left|2^{\ell}J\right|^{\frac{1}{n}(n+1-\alpha)}}\sqrt{\left|\widetilde{J}\right|_{\widetilde{\omega}}}\sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2+A_2^{\alpha,\mathrm{energy}}}\sqrt{\left|\widetilde{J}\right|_{\sigma}} \\
&\lesssim \boldsymbol{\tau}2^{-s}2^{-\ell}\sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2+A_2^{\alpha,\mathrm{energy}}}\sqrt{\frac{\left|\widetilde{J}\right|_{\widetilde{\omega}}}{\left|\widetilde{J}\right|^{1-\frac{\alpha}{n}}}\frac{\left|\widetilde{J}\right|_{\sigma}}{\left|\widetilde{J}\right|^{1-\frac{\alpha}{n}}}} \\
&\lesssim \boldsymbol{\tau}2^{-s}2^{-\ell}\sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2+A_2^{\alpha,\mathrm{energy}}}\sqrt{A_2^{\alpha,\mathrm{punct}}}\ .
\end{aligned}$$

These estimates are summable in both s and ℓ.

Now we turn to the terms $A_{s,close}^{\ell}(J)$, and recall from (9.24) that $A_{s,close}^{\ell}(J)=0$ if $\ell\geq 1+\delta s$. So we now suppose that $\ell\leq\delta s$. We have, with m as in (9.23),

$$\begin{aligned}
&A_{s,close}^{\ell}(J) \leq \sum_{F'\in\mathcal{F}} \sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*} \int_{2^{\ell-m}J} S_{(J',J)}^{F'}(y)\, d\sigma(y) \\
\approx\ & \sum_{F'\in\mathcal{F}} \sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*} \int_{2^{\ell-m}J} \frac{1}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{n+1-\alpha}} \frac{\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left|2^{\ell}J\right|^{\frac{1}{n}(n+1-\alpha)}} d\sigma(y) \\
=\ & \left(\sum_{F'\in\mathcal{F}} \sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*} \left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\right) \frac{1}{\left|2^{\ell}J\right|^{\frac{1}{n}(n+1-\alpha)}} \\
&\qquad\times \int_{2^{\ell-m}J} \frac{1}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{n+1-\alpha}} d\sigma(y).
\end{aligned}$$

Now we use the inequality $\sum_{F'\in\mathcal{F}}\left\|\mathsf{P}^{\omega}_{F',J'}\mathbf{z}\right\|^2_{L^2(\omega)}\leq\boldsymbol{\tau}\left|J'\right|^{\frac{2}{n}}\left|J'\right|_{\omega}$ and we get the relatively crude estimate

$$\begin{aligned}
&A^{\ell}_{s,close}(J)\\
\lesssim\;&\boldsymbol{\tau}2^{-2s}\left|J\right|^{\frac{2}{n}}\left|2^{\ell+2}J\setminus 2^{\ell-1}J\right|_{\omega}\frac{1}{\left|2^{\ell}J\right|^{\frac{1}{n}(n+1-\alpha)}}\int_{2^{\ell-m}J}\frac{1}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{n+1-\alpha}}d\sigma(y)\\
\lesssim\;&\boldsymbol{\tau}2^{-2s}\left|J\right|^{\frac{2}{n}}\frac{\left|2^{\ell+2}J\setminus 2^{\ell-1}J\right|_{\omega}}{\left|2^{\ell}J\right|^{\frac{1}{n}(n+1-\alpha)}}\frac{\left|2^{\ell-m}J\right|_{\sigma}}{\left|J\right|^{\frac{1}{n}(n+1-\alpha)}}\lesssim\boldsymbol{\tau}2^{-2s}\frac{\left|2^{\ell+2}J\setminus 2^{\ell-1}J\right|_{\omega}}{\left|2^{\ell+2}J\right|^{1-\frac{\alpha}{n}}}\frac{\left|2^{\ell-m}J\right|_{\sigma}}{\left|2^{\ell-m}J\right|^{1-\frac{\alpha}{n}}}2^{\ell(n-1-\alpha)}\\
\lesssim\;&\boldsymbol{\tau}2^{-2s}2^{\ell(n-1-\alpha)}A^{\alpha}_2\lesssim\boldsymbol{\tau}2^{-s}A^{\alpha}_2\ ,
\end{aligned}$$

provided that $\ell\leq\frac{s}{n}$ and $m>1$. But we are assuming $\ell\leq\delta s$ here, and so we obtain a suitable estimate for $A^{\ell}_{s,close}(J)$ from this crude estimate provided we choose $0<\delta<\frac{1}{n}$. Indeed, for fixed s we then have $\ell\leq\delta s<\frac{s}{n}$, and so also

$$2^{-2s}2^{\ell(n-1-\alpha)}\leq 2^{-2s}2^{s\left(\frac{n-1-\alpha}{n}\right)}\leq 2^{-s\left(\frac{n+1+\alpha}{n}\right)}\ ,$$

and hence

$$\sum_{l=1}^{\delta s}2^{-2s}2^{\ell(n-1-\alpha)}\leq C\frac{s}{n}2^{-s\left(\frac{n+1+\alpha}{n}\right)}\lesssim 2^{-s}\ .$$

The above estimates prove

$$T^{\mathrm{proximal}}_s+T^{\mathrm{difference}}_s\lesssim 2^{-s}\left(\mathcal{A}^{\alpha}_2+\sqrt{\left(\mathcal{E}^{\mathrm{plug}}_{\alpha}\right)^2+A^{\alpha,\mathrm{energy}}_2}\sqrt{A^{\alpha,\mathrm{punct}}_2}\right),$$

which is summable in s.

9.3.2. *The intersection term.* Now we return to the term,

$$\begin{aligned}
T^{\mathrm{intersection}}_s\;\equiv\;&\sum_{F,F'\in\mathcal{F}}\ \sum_{\substack{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(F),\ J'\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(F'\right)\\ J,J'\subset I,\ \ell\left(J'\right)=2^{-s}\ell(J)\\ \mathrm{qdist}(c_J,c_{J'})\geq 2^{s(1+\delta)}\ell\left(J'\right)}}\\
&\times\int_{B(J,J')}\frac{\left\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)}}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_J\right|\right)^{n+1-\alpha}}\frac{\left\|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\right\|^2_{L^2(\omega)}}{\left(\left|J'\right|^{\frac{1}{n}}+\left|y-c_{J'}\right|\right)^{n+1-\alpha}}d\sigma(y)\,.
\end{aligned}$$

It will suffice to show that $T^{\mathrm{intersection}}_s$ satisfies the estimate,

$$\begin{aligned}
T^{\mathrm{intersection}}_s\;\lesssim\;&2^{-s\delta}\sqrt{\left(\mathcal{E}^{\mathrm{plug}}_{\alpha}\right)^2+A^{\alpha,\mathrm{energy}}_2}\sqrt{A^{\alpha,\mathrm{punct}}_2}\sum_{F'\in\mathcal{F}'}\ \sum_{\substack{J'\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(F'\right)\\ J'\subset I}}\left\|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\right\|^2_{L^2(\omega)}\\
=\;&2^{-s\delta}\sqrt{\left(\mathcal{E}^{\mathrm{plug}}_{\alpha}\right)^2+A^{\alpha,\mathrm{energy}}_2}\sqrt{A^{\alpha,\mathrm{punct}}_2}\int_{\widehat{I}}t^2\overline{\mu}\ .
\end{aligned}$$

Using $B(J,J')=B\left(c_J,\frac{1}{2}\operatorname{qdist}(c_J,c_{J'})\right)$, we can write (suppressing some notation for clarity),

$$\begin{aligned}
&T_s^{\text{intersection}}\\
&= \sum_{F,F'}\sum_{J,J'}\int_{B(J,J')}\frac{\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(|J|^{\frac{1}{n}}+|y-c_J|\right)^{n+1-\alpha}}\frac{\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{\left(|J'|^{\frac{1}{n}}+|y-c_{J'}|\right)^{n+1-\alpha}}d\sigma(y)\\
&\approx \sum_{F,F'}\sum_{J,J'}\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\frac{1}{|c_J-c_{J'}|^{n+1-\alpha}}\int_{B(J,J')}\frac{d\sigma(y)}{\left(|J|^{\frac{1}{n}}+|y-c_J|\right)^{n+1-\alpha}}\\
&\leq \sum_{F'}\sum_{J'}\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\sum_{F}\sum_{J}\frac{1}{|c_J-c_{J'}|^{n+1-\alpha}}\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\int_{B(J,J')}\frac{d\sigma(y)}{\left(|J|^{\frac{1}{n}}+|y-c_J|\right)^{n+1-\alpha}}\\
&\equiv \sum_{F'}\sum_{J'}\left\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 S_s(J'),
\end{aligned}$$

and since $\int_{B(J,J')}\frac{d\sigma(y)}{\left(|J|^{\frac{1}{n}}+|y-c_J|\right)^{n+1-\alpha}}\approx\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{B(J,J')}\sigma\right)}{|J|^{\frac{1}{n}}}$, it remains to show that for each fixed J',

$$\begin{aligned}
S_s(J') &\approx \sum_{F}\sum_{J:\ \operatorname{qdist}(c_J,c_{J'})\geq 2^{s(1+\delta)}\ell(J')}^{*}\frac{\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2}{|c_J-c_{J'}|^{n+1-\alpha}}\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{B(J,J')}\sigma\right)}{|J|^{\frac{1}{n}}}\\
&\lesssim 2^{-\delta s}\sqrt{\left(\mathcal{E}_{\alpha}^{\text{plug}}\right)^2+A_2^{\alpha,\text{energy}}}\sqrt{A_2^{\alpha}}\ .
\end{aligned}$$

We write

$$\begin{aligned}
S_s(J') &\approx \sum_{k\geq s(1+\delta)}\frac{1}{\left(2^k|J'|^{\frac{1}{n}}\right)^{n+1-\alpha}}\sum_{F}\sum_{J:\ \operatorname{qdist}(c_J,c_{J'})\approx 2^k\ell(J')}^{*}\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{B(J,J')}\sigma\right)}{|J|^{\frac{1}{n}}}\\
&= \sum_{k\geq s(1+\delta)}\frac{1}{\left(2^k|J'|^{\frac{1}{n}}\right)^{n+1-\alpha}}S_s^k(J')\ ;\\
S_s^k(J') &\equiv \sum_{F}\sum_{J:\ \operatorname{qdist}(c_J,c_{J'})\approx 2^k\ell(J')}^{*}\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{B(J,J')}\sigma\right)}{|J|^{\frac{1}{n}}},
\end{aligned}\tag{9.28}$$

where by $\operatorname{qdist}(c_J,c_{J'})\approx 2^k\ell(J')$ we mean $2^k\ell(J')\leq\operatorname{qdist}(c_J,c_{J'})\leq 2^{k+1}\ell(J')$. Moreover, if $\operatorname{qdist}(c_J,c_{J'})\approx 2^k\ell(J')$, then from the fact that the quasiradius of $B(J,J')$ is $\frac{1}{2}\operatorname{qdist}(c_J,c_{J'})$, we obtain

$$B(J,J')\subset C_0 2^k J',$$

where C_0 is a positive constant ($C_0=6$ works).

For fixed $k\geq 1$, we invoke yet again the *'prepare to puncture'* argument. Choose an alternate quasicube $\widetilde{J'}\in\mathcal{A}\Omega\mathcal{D}$ such that $C_0 2^k J\subset\widetilde{J'}$ and $\ell\left(\widetilde{J'}\right)\leq C2^k\ell(J')$. Define $\widetilde{\omega}=\omega-\omega(\{p\})\delta_p$ where p is an atomic point in $\widetilde{J'}$ for which

$$\omega(\{p\})=\sup_{q\in\mathfrak{P}_{(\sigma,\omega)}:\ q\in\widetilde{J'}}\omega(\{q\}).$$

(If ω has no atomic point in common with σ in $\widetilde{J'}$ set $\widetilde{\omega}=\omega$.) Then we have $\left|\widetilde{J'}\right|_{\widetilde{\omega}}=\omega\left(\widetilde{J'},\mathfrak{P}_{(\sigma,\omega)}\right)$ and so from (9.17),

$$\frac{\left|\widetilde{J'}\right|_{\widetilde{\omega}}}{\left|\widetilde{J'}\right|^{1-\frac{\alpha}{n}}}\frac{\left|\widetilde{J'}\right|_{\sigma}}{\left|\widetilde{J'}\right|^{1-\frac{\alpha}{n}}}\leq A_2^{\alpha,\text{punct}}\text{ and }\sum_{F\in\mathcal{F}}\left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\lesssim\boldsymbol{\tau}\ell(J)^2|J|_{\widetilde{\omega}}\ .$$

Now we are ready to apply Cauchy-Schwarz and the alternate local estimate (9.16) in Lemma 18 with $M=\widetilde{J'}$ to the second line below to get the following estimate for $S_s^k(J')$ defined in (9.28) above:

$$\begin{aligned}
S_s^k(J') &\leq \left(\sum_F \sum_{J:\ \mathrm{qdist}(c_J,c_{J'})\approx 2^k\ell(J')} \left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\right)^{\frac{1}{2}} \\
&\quad\times\left(\sum_F \sum_{J:\ \mathrm{qdist}(c_J,c_{J'})\approx 2^k\ell(J')} \left\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{B(J,J')}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\right)^{\frac{1}{2}} \\
&\lesssim \left(\boldsymbol{\tau}2^{2s}|J'|^{\frac{2}{n}}\left|\widetilde{J'}\right|_{\widetilde{\omega}}\right)^{\frac{1}{2}}\left(\boldsymbol{\tau}\left[\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2+A_2^{\alpha,\mathrm{energy}}\right]\left|\widetilde{J'}\right|_{\sigma}\right)^{\frac{1}{2}} \\
&\lesssim \boldsymbol{\tau}\sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2+A_2^{\alpha,\mathrm{energy}}}\,2^s|J'|^{\frac{1}{n}}\sqrt{\left|\widetilde{J'}\right|_{\widetilde{\omega}}}\sqrt{\left|\widetilde{J'}\right|_{\sigma}} \\
&\lesssim \boldsymbol{\tau}\sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2+A_2^{\alpha,\mathrm{energy}}}\sqrt{A_2^{\alpha,\mathrm{punct}}}\,2^s|J'|^{\frac{1}{n}}\left|2^kJ'\right|^{1-\frac{\alpha}{n}} \\
&= \boldsymbol{\tau}\sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2+A_2^{\alpha,\mathrm{energy}}}\sqrt{A_2^{\alpha,\mathrm{punct}}}\,2^s2^{k(n-\alpha)}|J'|^{\frac{1}{n}(n+1-\alpha)}.
\end{aligned}$$

Altogether then we have

$$\begin{aligned}
S_s(J') &= \sum_{k\geq(1+\delta)s}\frac{1}{\left(2^k|J'|^{\frac{1}{n}}\right)^{n+1-\alpha}}S_s^k(J') \\
&\lesssim \sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2+A_2^{\alpha,\mathrm{energy}}}\sqrt{A_2^{\alpha,\mathrm{punct}}}\sum_{k\geq(1+\delta)s}\frac{1}{\left(2^k|J'|^{\frac{1}{n}}\right)^{n+1-\alpha}}2^s2^{k(n-\alpha)}|J'|^{\frac{1}{n}(n+1-\alpha)} \\
&\lesssim \sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2+A_2^{\alpha,\mathrm{energy}}}\sqrt{A_2^{\alpha,\mathrm{punct}}}\sum_{k\geq(1+\delta)s}2^{s-k}\lesssim 2^{-\delta s}\sqrt{\left(\mathcal{E}_{\alpha}^{\mathrm{plug}}\right)^2+A_2^{\alpha,\mathrm{energy}}}\sqrt{A_2^{\alpha,\mathrm{punct}}},
\end{aligned}$$

which is summable in s. This completes the proof of (9.21), and hence of the estimate for $\mathbf{Back}\left(\widehat{I}\right)$ in (9.18).

10. The stopping form

This section is virtually unchanged from the corresponding section in [SaShUr5]. In the one-dimensional setting of the Hilbert transform, Hytönen [Hyt2] observed that "...the innovative verification of the local estimate by Lacey [Lac] is already set up in such a way that it is ready for us to borrow as a black box." The same observation carries over in spirit here regarding the adaptation of Lacey's recursion and stopping time to proving the local estimate in [SaShUr5]. However, that adaptation involves the splitting of the stopping form into two sublinear forms, the first handled by methods in [LaSaUr2], and the second by methods in [Lac]. So for the convenience of the reader, we repeat all the details here, even though the arguments are little changed for common point masses.

In this section we adapt the argument of M. Lacey in [Lac] to apply in the setting of a general α-fractional Calderón-Zygmund operator T^{α} in $\mathbb{R}^n$ using the Monotonicity Lemma 7 and our quasienergy condition in Definition 8. We will prove the bound (7.16) for the stopping form

$$\begin{aligned}
\mathsf{B}_{\mathrm{stop}}^{A}(f,g) &\equiv \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}\\ J\Subset_{\boldsymbol{\rho},\varepsilon}I_J}}\left(\mathbb{E}_{I_J}^{\sigma}\bigtriangleup_I^{\sigma}f\right)\left\langle T_{\sigma}^{\alpha}\mathbf{1}_{A\setminus I_J},\bigtriangleup_J^{\omega}g\right\rangle_{\omega} \qquad (10.1)\\
&= \sum_{\substack{I:\ \pi I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}\\ J\Subset_{\boldsymbol{\rho},\varepsilon}I}}\left(\mathbb{E}_{I}^{\sigma}\bigtriangleup_{\pi I}^{\sigma}f\right)\left\langle T_{\sigma}^{\alpha}\mathbf{1}_{A\setminus I},\bigtriangleup_J^{\omega}g\right\rangle_{\omega},
\end{aligned}$$

where we have made the 'change of dummy variable' $I_J \to I$ for convenience in notation (recall that the child of I that contains J is denoted I_J).

However, the Monotonicity Lemma of Lacey and Wick has an additional term on the right hand side, and our quasienergy condition is not a direct generalization of the one-dimensional energy condition. These differences in higher dimension result in changes and complications that must be tracked throughout the argument. In particular, we find it necessary to separate the interaction of the two terms on the right side of the Monotonicity Lemma by splitting the stopping form into the two corresponding sublinear forms in (10.6) below. Recall that for $A \in \mathcal{A}$ the *shifted* corona is given in Definition 13 by

$$\mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}} = \{J \in \mathcal{C}_A : J \Subset_{\boldsymbol{\tau},\varepsilon} A\} \cup \bigcup_{A' \in \mathfrak{C}_{\mathcal{A}}(A)} \{J \in \Omega\mathcal{D} : J \Subset_{\boldsymbol{\tau},\varepsilon} A \text{ and } J \text{ is } \boldsymbol{\tau}\text{-nearby in } A'\},$$

and in particular the $\mathbf{1}$-shifted corona is given by $\mathcal{C}_A^{\mathbf{1}-\mathrm{shift}} = (\mathcal{C}_A \setminus \{A\}) \cup \mathfrak{C}_{\mathcal{A}}(A)$.

Definition 17. *Suppose that $A \in \mathcal{A}$ and that $\mathcal{P} \subset \mathcal{C}_A^{\mathbf{1}-\mathrm{shift}} \times \mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}$. We say that the collection of pairs $\mathcal{P}$ is* A-admissible *if*

- (good and $(\boldsymbol{\rho}-\mathbf{1},\varepsilon)$-deeply embedded) J is good and $J \Subset_{\boldsymbol{\rho}-\mathbf{1},\varepsilon} I \subsetneqq A$ for every $(I,J) \in \mathcal{P}$,
- (tree-connected in the first component) if $I_1 \subset I_2$ and both $(I_1,J) \in \mathcal{P}$ and $(I_2,J) \in \mathcal{P}$, then $(I,J) \in \mathcal{P}$ for every I in the geodesic $[I_1,I_2] = \{I \in \Omega\mathcal{D} : I_1 \subset I \subset I_2\}$.

However, since $(I,J) \in \mathcal{P}$ implies both $J \in \mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}$ and $J \Subset_{\boldsymbol{\rho}-\mathbf{1},\varepsilon} I$, the assumption $\boldsymbol{\rho} > \boldsymbol{\tau}$ in Definition 12 shows that I is in the corona $\mathcal{C}_A$, and hence we may replace $\mathcal{C}_A^{\mathbf{1}-\mathrm{shift}}$ with the restricted corona $\mathcal{C}'_A \equiv \mathcal{C}_A \setminus \{A\}$ in the above definition of A-*admissible*. The basic example of an admissible collection of pairs is obtained from the pairs of quasicubes summed in the stopping form $\mathsf{B}_{stop}^A(f,g)$ in (10.1), which occurs in (7.16) above;

$$\mathcal{P}^A \equiv \left\{(I,J) : I \in \mathcal{C}'_A \text{ and } J \in \mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}} \text{ where } J \text{ is } \boldsymbol{\tau}\text{-good and } J \Subset_{\boldsymbol{\rho}-\mathbf{1},\varepsilon} I\right\}. \tag{10.2}$$

Recall also that J is $\boldsymbol{\tau}$-good if $J \in \Omega\mathcal{D}^{\boldsymbol{\tau}}_{(\mathbf{r},\varepsilon)-\mathrm{good}}$ as in (3.11), i.e. if J and its children and its ℓ-parents up to level $\boldsymbol{\tau}$ are all good. Recall that the quasiHaar support of g is contained in the collection of $\boldsymbol{\tau}$-good quasicubes.

Definition 18. *Suppose that $A \in \mathcal{A}$ and that $\mathcal{P}$ is an* A-admissible *collection of pairs. Define the associated* stopping *form* $\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}}$ *by*

$$\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}}(f,g) \equiv \sum_{(I,J)\in\mathcal{P}} \left(\mathbb{E}_I^\sigma \triangle_{\pi I}^\sigma f\right) \left\langle T_\sigma^\alpha \mathbf{1}_{A\setminus I}, \triangle_J^\omega g\right\rangle_\omega ,$$

where we may of course further restrict I to $\pi I \in \operatorname{supp}\widehat{f}$ if we wish.

Given an A-admissible collection $\mathcal{P}$ of pairs define the reduced collection $\mathcal{P}^{\mathrm{red}}$ as follows. For each fixed J let I_J^{red} be the largest good quasicube I such that $(I,J) \in \mathcal{P}$. Then set

$$\mathcal{P}^{\mathrm{red}} \equiv \left\{(I,J) \in \mathcal{P} : I \subset I_J^{\mathrm{red}}\right\}.$$

Clearly $\mathcal{P}^{\mathrm{red}}$ is A-admissible. Now recall our assumption that the quasiHaar support of f is contained in the set of $\boldsymbol{\tau}$-good quasicubes, which in particular requires that their children are all good as well. This assumption has the important implication that $\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}}(f,g) = \mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}^{\mathrm{red}}}(f,g)$. Indeed, if $(I,J) \in \mathcal{P} \setminus \mathcal{P}^{\mathrm{red}}$ then $\pi I \notin \operatorname{supp}\widehat{f}$ and so $\mathbb{E}_I^\sigma \triangle_{\pi I}^\sigma f = 0$. Thus for the purpose of bounding the stopping form, we may assume that the following additional property holds for any A-admissible collection of pairs $\mathcal{P}$:

- if $(I,J) \in \mathcal{P}$ is maximal in the sense that $I \supset I'$ for all I' satisfying $(I',J) \in \mathcal{P}$, then I is good.

Note that there is an asymmetry in our definition of $\mathcal{P}^{\mathrm{red}}$ here, namely that the second components J are required to be $\boldsymbol{\tau}$-good, while the maximal first components I are required to be good. Of course the treatment of the dual stopping forms will use the reversed requirements, and this accounts for our symmetric restrictions imposed on the quasiHaar supports of f and g at the outset of the proof.

Definition 19. *We say that an admissible collection $\mathcal{P}$ is* reduced *if $\mathcal{P} = \mathcal{P}^{\mathrm{red}}$, so that the additional property above holds.*

Note that $\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}}(f,g)=\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}}\left(\mathsf{P}_{\mathcal{C}_A}^{\sigma}f,\mathsf{P}_{\mathcal{C}_A^{\tau-\mathrm{shift}}}^{\omega}g\right)$. Recall that the deep quasienergy condition constant $\mathcal{E}_{\alpha}^{\mathrm{deep}}$ is given by

$$\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}\right)^2\equiv\sup_{I=\dot{\cup}I_r}\frac{1}{|I|_{\sigma}}\sum_{r=1}^{\infty}\sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I_r)}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I\setminus\gamma J}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 .$$

Proposition 3. *Suppose that $A\in\mathcal{A}$ and that $\mathcal{P}$ is an* A-admissible *collection of pairs. Then the stopping form $\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}}$ satisfies the bound*

$$\left|\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}}(f,g)\right|\lesssim\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}+\sqrt{A_2^{\alpha}}\right)\left(\|f\|_{L^2(\sigma)}+\alpha_{\mathcal{A}}(f)\sqrt{|A|_{\sigma}}\right)\|g\|_{L^2(\omega)} . \tag{10.3}$$

With this proposition in hand, we can complete the proof of (7.16), and hence of Theorem 1, by summing over the stopping quasicubes $A\in\mathcal{A}$ with the choice $\mathcal{P}^A$ of A-admissible pairs for each A:

$$\begin{aligned}
&\sum_{A\in\mathcal{A}}\left|\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}^A}(f,g)\right|\\
\lesssim\ &\sum_{A\in\mathcal{A}}\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}+\sqrt{A_2^{\alpha}}\right)\left(\|\mathsf{P}_{\mathcal{C}_A}f\|_{L^2(\sigma)}+\alpha_{\mathcal{A}}(f)\sqrt{|A|_{\sigma}}\right)\left\|\mathsf{P}_{\mathcal{C}_A^{\tau-\mathrm{shift}}}g\right\|_{L^2(\omega)}\\
\lesssim\ &\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}+\sqrt{A_2^{\alpha}}\right)\left(\sum_{A\in\mathcal{A}}\left(\|\mathsf{P}_{\mathcal{C}_A}f\|_{L^2(\sigma)}^2+\alpha_{\mathcal{A}}(f)^2|A|_{\sigma}\right)\right)^{\frac{1}{2}}\left(\sum_{A\in\mathcal{A}}\left\|\mathsf{P}_{\mathcal{C}_A^{\tau-\mathrm{shift}}}g\right\|_{L^2(\omega)}^2\right)^{\frac{1}{2}}\\
\lesssim\ &\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}+\sqrt{A_2^{\alpha}}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)} ,
\end{aligned}$$

by orthogonality $\sum_{A\in\mathcal{A}}\|\mathsf{P}_{\mathcal{C}_A}f\|_{L^2(\sigma)}^2\leq\|f\|_{L^2(\sigma)}^2$ in corona projections $\mathcal{C}_A^{\sigma}$, 'quasi' orthogonality $\sum_{A\in\mathcal{A}}\alpha_{\mathcal{A}}(f)^2|A|_{\sigma}\lesssim\|f\|_{L^2(\sigma)}^2$ in the stopping quasicubes $\mathcal{A}$, and by the bounded overlap of the shifted coronas $\mathcal{C}_A^{\tau-\mathrm{shift}}$:

$$\sum_{A\in\mathcal{A}}\mathbf{1}_{\mathcal{C}_A^{\tau-\mathrm{shift}}}\leq\boldsymbol{\tau}\mathbf{1}_{\Omega\mathcal{D}}.$$

To prove Proposition 3, we begin by letting $\Pi_2\mathcal{P}$ consist of the second components of the pairs in $\mathcal{P}$ and writing

$$\begin{aligned}
\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}}(f,g)&=\sum_{J\in\Pi_2\mathcal{P}}\left\langle T_{\sigma}^{\alpha}\varphi_J^{\mathcal{P}},\triangle_J^{\omega}g\right\rangle_{\omega};\\
\text{where }\varphi_J^{\mathcal{P}}&\equiv\sum_{I\in\mathcal{C}_A':\ (I,J)\in\mathcal{P}}\mathbb{E}_I^{\sigma}\left(\triangle_{\pi I}^{\sigma}f\right)\ \mathbf{1}_{A\setminus I} .
\end{aligned}$$

By the tree-connected property of $\mathcal{P}$, and the telescoping property of martingale differences, together with the bound $\alpha_{\mathcal{A}}(A)$ on the quasiaverages of f in the corona $\mathcal{C}_A$, we have

$$\left|\varphi_J^{\mathcal{P}}\right|\lesssim\alpha_{\mathcal{A}}(A)\,1_{A\setminus I_{\mathcal{P}}(J)}, \tag{10.4}$$

where $I_{\mathcal{P}}(J)\equiv\bigcap\{I:(I,J)\in\mathcal{P}\}$ is the smallest quasicube I for which $(I,J)\in\mathcal{P}$. Another important property of these functions is the sublinearity:

$$\left|\varphi_J^{\mathcal{P}}\right|\leq\left|\varphi_J^{\mathcal{P}_1}\right|+\left|\varphi_J^{\mathcal{P}_2}\right|,\qquad\mathcal{P}=\mathcal{P}_1\dot{\cup}\mathcal{P}_2 . \tag{10.5}$$

Now apply the Monotonicity Lemma 7 to the inner product $\left\langle T_{\sigma}^{\alpha}\varphi_J,\triangle_J^{\omega}g\right\rangle_{\omega}$ to obtain

$$\begin{aligned}
\left|\left\langle T_{\sigma}^{\alpha}\varphi_J,\triangle_J^{\omega}g\right\rangle_{\omega}\right|&\lesssim\frac{\mathrm{P}^{\alpha}\left(J,|\varphi_J|\,\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma\right)}{|J|^{\frac{1}{n}}}\left\|\triangle_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}\left\|\triangle_J^{\omega}g\right\|_{L^2(\omega)}\\
&\quad+\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J,|\varphi_J|\,\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma\right)}{|J|^{\frac{1}{n}}}\left\|\mathsf{P}_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}\left\|\triangle_J^{\omega}g\right\|_{L^2(\omega)} .
\end{aligned}$$

Thus we have

(10.6)

$$\begin{aligned}\left|\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}}\left(f,g\right)\right| &\leq \sum_{J\in\Pi_2\mathcal{P}}\frac{\mathrm{P}_1^{\alpha}\left(J,\left|\varphi_J\right|\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\left\|\bigtriangleup_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}\left\|\bigtriangleup_J^{\omega}g\right\|_{L^2(\omega)}\\&\quad+\sum_{J\in\Pi_2\mathcal{P}}\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J,\left|\varphi_J\right|\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\left\|\mathsf{P}_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}\left\|\bigtriangleup_J^{\omega}g\right\|_{L^2(\omega)}\\&\equiv \left|\mathsf{B}\right|_{\mathrm{stop},1,\bigtriangleup^{\omega}}^{A,\mathcal{P}}\left(f,g\right)+\left|\mathsf{B}\right|_{\mathrm{stop},1+\delta,\mathsf{P}^{\omega}}^{A,\mathcal{P}}\left(f,g\right),\end{aligned}$$

where we have dominated the stopping form by two sublinear stopping forms that involve the Poisson integrals of order 1 and $1+\delta$ respectively, and where the smaller Poisson integral $\mathrm{P}_{1+\delta}^{\alpha}$ is multiplied by the larger projection $\left\|\mathsf{P}_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}$. This splitting turns out to be successful in separating the two energy terms from the right hand side of the Energy Lemma, because of the two properties (10.4) and (10.5) above. It remains to show the two inequalities:

$$\left|\mathsf{B}\right|_{\mathrm{stop},1,\bigtriangleup^{\omega}}^{A,\mathcal{P}}\left(f,g\right)\lesssim\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}+\sqrt{A_2^{\alpha}}\right)\alpha_{\mathcal{A}}\left(A\right)\sqrt{\left|A\right|_{\sigma}}\left\|g\right\|_{L^2(\omega)},\tag{10.7}$$

for $f\in L^2\left(\sigma\right)$ satisfying where $\mathbb{E}_I^{\sigma}\left|f\right|\leq\alpha_{\mathcal{A}}\left(A\right)$ for all $I\in\mathcal{C}_A$; and

$$\left|\mathsf{B}\right|_{\mathrm{stop},1+\delta,\mathsf{P}^{\omega}}^{A,\mathcal{P}}\left(f,g\right)\lesssim\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}+\sqrt{A_2^{\alpha}}\right)\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)},\tag{10.8}$$

where we only need the case $\mathcal{P}=\mathcal{P}^A$ in this latter inequality as there is no recursion involved in treating this second sublinear form. We consider first the easier inequality (10.8) that does not require recursion. In the subsequent subsections we will control the more difficult inequality (10.7) by adapting the stopping time and recursion of M. Lacey to the sublinear form $\left|\mathsf{B}\right|_{\mathrm{stop},1,\bigtriangleup^{\omega}}^{A,\mathcal{P}}\left(f,g\right)$.

10.1. **The second inequality.** Now we turn to proving (10.8), i.e.

$$\left|\mathsf{B}\right|_{\mathrm{stop},1+\delta,\mathsf{P}^{\omega}}^{A,\mathcal{P}}\left(f,g\right)\lesssim\left(\mathcal{E}_{\alpha}^{\mathrm{deep}}+\sqrt{A_2^{\alpha}}\right)\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)},$$

where since

$$\left|\varphi_J\right|=\left|\sum_{I\in\mathcal{C}_A':\ (I,J)\in\mathcal{P}}\mathbb{E}_I^{\sigma}\left(\bigtriangleup_{\pi I}^{\sigma}f\right)\ \mathbf{1}_{A\setminus I}\right|\leq\sum_{I\in\mathcal{C}_A':\ (I,J)\in\mathcal{P}}\left|\mathbb{E}_I^{\sigma}\left(\bigtriangleup_{\pi I}^{\sigma}f\right)\ \mathbf{1}_{A\setminus I}\right|,$$

the sublinear form $\left|\mathsf{B}\right|_{\mathrm{stop},1+\delta,\mathsf{P}^{\omega}}^{A,\mathcal{P}}$ can be dominated and then decomposed by pigeonholing the ratio of side lengths of J and I:

$$\begin{aligned}\left|\mathsf{B}\right|_{\mathrm{stop},1+\delta,\mathsf{P}^{\omega}}^{A,\mathcal{P}}\left(f,g\right) &= \sum_{J\in\Pi_2\mathcal{P}}\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J,\left|\varphi_J\right|\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\left\|\mathsf{P}_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}\left\|\bigtriangleup_J^{\omega}g\right\|_{L^2(\omega)}\\&\leq \sum_{(I,J)\in\mathcal{P}}\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J,\left|\mathbb{E}_I^{\sigma}\left(\bigtriangleup_{\pi I}^{\sigma}f\right)\right|\mathbf{1}_{A\setminus I}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\left\|\mathsf{P}_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}\left\|\bigtriangleup_J^{\omega}g\right\|_{L^2(\omega)}\\&\equiv \sum_{s=0}^{\infty}\left|\mathsf{B}\right|_{\mathrm{stop},1+\delta,\mathsf{P}^{\omega}}^{A,\mathcal{P};s}\left(f,g\right);\\\left|\mathsf{B}\right|_{\mathrm{stop},1+\delta,\mathsf{P}^{\omega}}^{A,\mathcal{P};s}\left(f,g\right) &\equiv \sum_{\substack{(I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J,\left|\mathbb{E}_I^{\sigma}\left(\bigtriangleup_{\pi I}^{\sigma}f\right)\right|\mathbf{1}_{A\setminus I}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\left\|\mathsf{P}_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}\left\|\bigtriangleup_J^{\omega}g\right\|_{L^2(\omega)}.\end{aligned}$$

Here we have the *entire* projection $\mathsf{P}_J^{\omega}\mathbf{x}$ onto all of the dyadic subquasicubes of J, but this is offset by the smaller Poisson integral $\mathrm{P}_{1+\delta}^{\alpha}$. We will now adapt the argument for the stopping term starting on page 42 of [LaSaUr2], where the geometric gain from the assumed Energy Hypothesis there will be replaced by a geometric gain from the smaller Poisson integral $\mathrm{P}_{1+\delta}^{\alpha}$ used here.

First, we exploit the additional decay in the Poisson integral $\mathrm{P}^{\alpha}_{1+\delta}$ as follows. Suppose that $(I,J)\in\mathcal{P}$ with $\ell(J)=2^{-s}\ell(I)$. We then compute

$$\begin{aligned}
\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}} &\approx \int_{A\setminus I}\frac{|J|^{\frac{\delta}{n}}}{|y-c_J|^{n+1+\delta-\alpha}}d\sigma(y)\\
&\leq \int_{A\setminus I}\left(\frac{|J|^{\frac{1}{n}}}{\operatorname{qdist}(c_J,I^c)}\right)^{\delta}\frac{1}{|y-c_J|^{n+1-\alpha}}d\sigma(y)\\
&\lesssim \left(\frac{|J|^{\frac{1}{n}}}{\operatorname{qdist}(c_J,I^c)}\right)^{\delta}\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}},
\end{aligned}$$

and use the goodness inequality,

$$\operatorname{qdist}(c_J,I^c)\geq\frac{1}{2}\ell(I)^{1-\varepsilon}\,\ell(J)^{\varepsilon}\geq\frac{1}{2}2^{s(1-\varepsilon)}\ell(J),$$

to conclude that

$$\left(\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}\right)\lesssim 2^{-s\delta(1-\varepsilon)}\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}. \tag{10.9}$$

We next claim that for $s\geq 0$ an integer,

$$\begin{aligned}
|\mathsf{B}|^{A,\mathcal{P};s}_{\mathrm{stop},1+\delta,\mathsf{P}^{\omega}}(f,g) &= \sum_{\substack{(I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\left|\mathbb{E}^{\sigma}_{I}\left(\triangle^{\sigma}_{\pi I}f\right)\right|\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}\left\|\mathsf{P}^{\omega}_{J}\mathbf{x}\right\|_{L^2(\omega)}\left\|\triangle^{\omega}_{J}g\right\|_{L^2(\omega)}\\
&\lesssim 2^{-s\delta(1-\varepsilon)}\left(\mathcal{E}^{\mathrm{deep}}_{\alpha}+\sqrt{A^{\alpha}_{2}}\right)\|f\|_{L^2(\sigma)}\ \|g\|_{L^2(\omega)}\ ,
\end{aligned}$$

from which (10.8) follows upon summing in $s\geq 0$. Now using both

$$\begin{aligned}
\left|\mathbb{E}^{\sigma}_{I}\left(\triangle^{\sigma}_{\pi I}f\right)\right| &= \frac{1}{|I|_{\sigma}}\int_{I}\left|\triangle^{\sigma}_{\pi I}f\right|d\sigma\leq\left\|\triangle^{\sigma}_{\pi I}f\right\|_{L^2(\sigma)}\frac{1}{\sqrt{|I|_{\sigma}}},\\
2^n\|f\|^2_{L^2(\sigma)} &= \sum_{I\in\Omega\mathcal{D}}\left\|\triangle^{\sigma}_{\pi I}f\right\|^2_{L^2(\sigma)}\ ,
\end{aligned}$$

we apply Cauchy-Schwarz in the I variable above to see that

$$\begin{aligned}
&\left[|\mathsf{B}|^{A,\mathcal{P};s}_{\mathrm{stop},1+\delta,\mathsf{P}^{\omega}}(f,g)\right]^2\lesssim\|f\|^2_{L^2(\sigma)}\\
&\times\left[\sum_{I\in\mathcal{C}'_A}\left(\frac{1}{\sqrt{|I|_{\sigma}}}\sum_{\substack{J:\ (I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}\left\|\mathsf{P}^{\omega}_{J}\mathbf{x}\right\|_{L^2(\omega)}\left\|\triangle^{\omega}_{J}g\right\|_{L^2(\omega)}\right)^2\right]^{\frac{1}{2}}.
\end{aligned}$$

We can then estimate the sum inside the square brackets by

$$\sum_{I\in\mathcal{C}'_A}\left\{\sum_{\substack{J:\ (I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\left\|\triangle^{\omega}_{J}g\right\|^2_{L^2(\omega)}\right\}\sum_{\substack{J:\ (I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\frac{1}{|I|_{\sigma}}\left(\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}^{\omega}_{J}\mathbf{x}\right\|^2_{L^2(\omega)}\lesssim\|g\|^2_{L^2(\omega)}A(s)^2,$$

where

$$A(s)^2\equiv\sup_{I\in\mathcal{C}'_A}\sum_{\substack{J:\ (I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\frac{1}{|I|_{\sigma}}\left(\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|\mathsf{P}^{\omega}_{J}\mathbf{x}\right\|^2_{L^2(\omega)}\ .$$

Finally then we turn to the analysis of the supremum in last display. From the Poisson decay (10.9) we have

$$\begin{aligned}
A(s)^2 &\lesssim \sup_{I\in\mathcal{C}'_A} \frac{1}{|I|_\sigma} 2^{-2s\delta(1-\varepsilon)} \sum_{\substack{J:\ (I,J)\in\mathcal{P} \\ \ell(J)=2^{-s}\ell(I)}} \left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \|\mathsf{P}^\omega_J x\|^2_{L^2(\omega)} \\
&\lesssim \sup_{I\in\mathcal{C}'_A} \frac{1}{|I|_\sigma} 2^{-2s\delta(1-\varepsilon)} \sum_{K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I)} \left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus I}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \sum_{\substack{J\subset K:\ (I,J)\in\mathcal{P} \\ \ell(J)=2^{-s}\ell(I)}} \|\mathsf{P}^\omega_J x\|^2_{L^2(\omega)} \\
&\lesssim 2^{-2s\delta(1-\varepsilon)} \left[\left(\mathcal{E}^{\mathrm{deep}}_\alpha\right)^2 + A_2^\alpha\right],
\end{aligned}$$

where the last inequality is the one for which the definition of quasienergy stopping quasicubes was designed. Indeed, from Definition 9, as $(I,J)\in\mathcal{P}$, we have that I is *not* a stopping quasicube in $\mathcal{A}$, and hence that (7.1) *fails* to hold, delivering the estimate above since $J \Subset_{\boldsymbol{\rho}-\mathbf{1},\varepsilon} I$ good must be contained in some $K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I)$, and since $\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}} \approx \frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus I}\sigma\right)}{|K|^{\frac{1}{n}}}$. The terms $\|\mathsf{P}^\omega_J x\|^2_{L^2(\omega)}$ are additive since the $J's$ are pigeonholed by $\ell(J)=2^{-s}\ell(I)$.

10.2. **The first inequality and the recursion of M. Lacey.** Now we turn to proving the more difficult inequality (10.7). Recall that in dimension $n=1$ the energy condition

$$\sum_{n=1}^\infty |J_n|_\omega \,\mathsf{E}(J_n,\omega)^2\, \mathrm{P}(J_n,\mathbf{1}_I\sigma)^2 \lesssim (\mathcal{NTV})\ |I|_\sigma, \qquad \dot{\bigcup}_{n=1}^\infty J_n \subset I,$$

could not be used in the NTV argument, because the set functional $J\to |J|_\omega \,\mathsf{E}(J,\omega)^2$ failed to be superadditive. On the other hand, the pivotal condition of NTV,

$$\sum_{n=1}^\infty |J_n|_\omega \,\mathrm{P}(J_n,\mathbf{1}_I\sigma)^2 \lesssim |I|_\sigma, \qquad \dot{\bigcup}_{n=1}^\infty J_n \subset I,$$

succeeded in the NTV argument because the set functional $J\to |J|_\omega$ is trivially superadditive, indeed additive. The final piece of the argument needed to prove the NTV conjecture was found by M. Lacey in [Lac], and amounts to first replacing the additivity of the functional $J\to |J|_\omega$ with the additivity of the projection functional $\mathcal{H}\to \|\mathsf{P}^\omega_{\mathcal{H}} x\|^2_{L^2(\omega)}$ defined on subsets $\mathcal{H}$ of the dyadic quasigrid $\Omega\mathcal{D}$. Then a stopping time argument relative to this more subtle functional, together with a clever recursion, constitute the main new ingredients in Lacey's argument [Lac].

To begin the extension to a more general Calderón-Zygmund operator T^α, we also recall the stopping quasienergy generalized to higher dimensions by

$$\mathbf{X}^\alpha(\mathcal{C}_A)^2 \equiv \sup_{I\in\mathcal{C}_A} \frac{1}{|I|_\sigma} \sum_{J\in\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I)} \left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{A\setminus\gamma J}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\mathrm{subgood},\omega}_J \mathbf{x}\right\|^2_{L^2(\omega)},$$

where $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I)$ is the set of maximal $\mathbf{r}$-deeply embedded subquasicubes of I where $\mathbf{r}$ is the goodness parameter. What now follows is an adaptation to our deep quasienergy condition and the sublinear form $|\mathsf{B}|^{A,\mathcal{P}}_{\mathrm{stop},1,\triangle^\omega}$ of the arguments of M. Lacey in [Lac]. We have the following Poisson inequality for quasicubes $B\subset A\subset I$:

$$\begin{aligned}
(10.10)\qquad \frac{\mathrm{P}^\alpha\left(A,\mathbf{1}_{I\setminus A}\sigma\right)}{|A|^{\frac{1}{n}}} &\approx \int_{I\setminus A} \frac{1}{\left(|y-c_A|\right)^{n+1-\alpha}} d\sigma(y) \\
&\lesssim \int_{I\setminus A} \frac{1}{\left(|y-c_B|\right)^{n+1-\alpha}} d\sigma(y) \approx \frac{\mathrm{P}^\alpha\left(B,\mathbf{1}_{I\setminus A}\sigma\right)}{|B|^{\frac{1}{n}}}.
\end{aligned}$$

10.3. **The stopping energy.** Fix $A \in \mathcal{A}$. We will use a 'decoupled' modification of the stopping energy $\mathbf{X}(\mathcal{C}_A)$. Suppose that $\mathcal{P}$ is an A-admissible collection of pairs of quasicubes in the product set $\Omega\mathcal{D} \times \Omega\mathcal{D}_{\mathrm{good}}$ of pairs of dyadic quasicubes in $\mathbb{R}^n$ with second component good. For an admissible collection $\mathcal{P}$ let $\Pi_1\mathcal{P}$ and $\Pi_2\mathcal{P}$ be the quasicubes in the first and second components of the pairs in $\mathcal{P}$ respectively, let $\Pi\mathcal{P} \equiv \Pi_1\mathcal{P} \cup \Pi_2\mathcal{P}$, and for $K \in \Pi\mathcal{P}$ define the $\boldsymbol{\tau}$-deep projection of $\mathcal{P}$ relative to K by

$$\Pi_2^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P} \equiv \left\{J \in \Pi_2\mathcal{P} : \ J \Subset_{\boldsymbol{\tau},\varepsilon} K\right\}.$$

Now the quasicubes J in $\Pi_2\mathcal{P}$ are of course *always* good, but this is *not* the case for quasicubes I in $\Pi_1\mathcal{P}$. Indeed, the collection $\mathcal{P}$ is tree-connected in the first component, and it is clear that there can be many *bad* quasicubes in a connected geodesic in the tree $\Omega\mathcal{D}$. But the quasiHaar support of f is contained in *good* quasicubes I, and we have also assumed that the children of these quasicubes I are good as well. As a consequence we may always assume that our A-admissible collections $\mathcal{P}$ are reduced in the sense of Definition 19. Thus we will use as our 'size testing collection' of quasicubes for $\mathcal{P}$ the collection

$$\Pi^{\mathrm{goodbelow}}\mathcal{P} \equiv \left\{K' \in \Omega\mathcal{D} : K' \text{ is good and } K' \subset K \text{ for some } K \in \Pi\mathcal{P}\right\},$$

which consists of all the good subquasicubes of any quasicube in $\Pi\mathcal{P}$. Note that the maximal quasicubes in $\Pi\mathcal{P} = \Pi\mathcal{P}^{\mathrm{red}}$ are already good themselves, and so we have the important property that

$$(I,J) \in \mathcal{P} = \mathcal{P}^{\mathrm{red}} \text{ implies } I \subset K \text{ for some quasicube } K \in \Pi^{\mathrm{goodbelow}}\mathcal{P}. \tag{10.11}$$

Now define the 'size functional' $\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})$ of $\mathcal{P}$ as follows. Recall that a projection $\mathsf{P}_{\mathcal{H}}^{\omega}$ on $\mathbf{x}$ satisfies

$$\left\|\mathsf{P}_{\mathcal{H}}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 = \sum_{J \in \mathcal{H}} \left\|\triangle_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 .$$

Definition 20. *If $\mathcal{P}$ is A-admissible, define*

$$\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})^2 \equiv \sup_{K \in \Pi^{\mathrm{goodbelow}}\mathcal{P}} \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_{\Pi_2^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P}}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 . \tag{10.12}$$

We should remark that that the quasicubes K in $\Pi^{\mathrm{goodbelow}}\mathcal{P}$ that fail to contain any $\boldsymbol{\tau}$-parents of quasicubes from $\Pi_2\mathcal{P}$ will not contribute to the size functional since $\Pi_2^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P}$ is empty in this case. We note three essential properties of this definition of size functional:

1. **Monotonicity** of size: $\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P}) \leq \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{Q})$ if $\mathcal{P} \subset \mathcal{Q}$,
2. **Goodness** of testing quasicubes: $\Pi^{\mathrm{goodbelow}}\mathcal{P} \subset \Omega\mathcal{D}_{\mathrm{good}}$,
3. **Control** by deep quasienergy condition: $\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P}) \lesssim \mathcal{E}_\alpha^{\mathrm{deep}} + \sqrt{A_2^\alpha}$.

The monotonicity property follows from $\Pi^{\mathrm{goodbelow}}\mathcal{P} \subset \Pi^{\mathrm{goodbelow}}\mathcal{Q}$ and $\Pi_2^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P} \subset \Pi_2^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{Q}$, and the goodness property follows from the definition of $\Pi^{\mathrm{goodbelow}}\mathcal{P}$. The control property is contained in the next lemma, which uses the stopping quasienergy control for the form $\mathsf{B}_{stop}^A(f,g)$ associated with A.

Lemma 19. *If $\mathcal{P}^A$ is as in (10.2) and $\mathcal{P} \subset \mathcal{P}^A$, then*

$$\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P}) \leq \mathbf{X}_\alpha(\mathcal{C}_A) \lesssim \mathcal{E}_\alpha^{\mathrm{deep}} + \sqrt{A_2^\alpha} + \sqrt{A_2^{\alpha,\mathrm{punct}}} .$$

Proof. Suppose that $K \in \Pi^{\mathrm{goodbelow}}\mathcal{P}$. To prove the first inequality in the statement we note that

$$\begin{aligned}
& \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_{\left(\Pi_2^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P}\right)^*}^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\
\leq\ & \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \sum_{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)} \left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\
\lesssim\ & \frac{1}{|K|_\sigma} \sum_{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{A\setminus K}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\
\lesssim\ & \frac{1}{|K|_\sigma} \sum_{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)} \left(\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{A\setminus \gamma J}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \leq \mathbf{X}_\alpha(\mathcal{C}_A)^2 ,
\end{aligned}$$

where the first inequality above follows since every $J' \in \Pi_2^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P}$ is contained in some $J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(I)$, the second inequality follows from (10.10) with $J \subset K \subset A$, and then the third inequality follows since $J \Subset_{\mathbf{r},\varepsilon} I$ implies $\gamma J \subset I$ by (3.2), and finally since $\Pi_2^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P} = \emptyset$ if $K \subset A$ and $K \notin \mathcal{C}_A$ by (10.13) below. The second inequality in the statement of the lemma follows from (7.5). □

The following useful fact is needed above and will be used later as well:

$$K \subset A \text{ and } K \notin \mathcal{C}_A \Longrightarrow \Pi_2^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P} = \emptyset \ . \tag{10.13}$$

To see this, suppose that $K \in \mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}} \setminus \mathcal{C}_A$. Then $K \subset A'$ for some $A' \in \mathfrak{C}_{\mathcal{A}}(A)$, and so if there is $J \in \Pi_2^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P}$, then $\ell(J) \leq 2^{-\boldsymbol{\tau}}\ell(K) \leq 2^{-\boldsymbol{\tau}}\ell(A')$, which implies that $J \notin \mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}$, which contradicts $\Pi_2^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P} \subset \mathcal{C}_A^{\boldsymbol{\tau}-\mathrm{shift}}$.

We remind the reader again that $c\,|J|^{\frac{1}{n}} \leq \ell(J) \leq C\,|J|^{\frac{1}{n}}$ for any quasicube J, and that we will generally use $|J|^{\frac{1}{n}}$ in the Poisson integrals and estimates, but will usually use $\ell(J)$ when defining collections of quasicubes. Now define an atomic measure $\omega_{\mathcal{P}}$ in the upper half space $\mathbb{R}_+^{n+1}$ by

$$\omega_{\mathcal{P}} \equiv \sum_{J \in \Pi_2\mathcal{P}} \left\|\triangle_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \ \delta_{(c_J,\ell(J))}.$$

Define the tent $\mathbf{T}(K)$ over a quasicube $K = \Omega L$ to be $\Omega(\mathbf{T}(L))$ where $\mathbf{T}(L)$ is the convex hull of the n-cube $L \times \{0\}$ and the point $(c_L, \ell(L)) \in \mathbb{R}_+^{n+1}$. Define the $\boldsymbol{\tau}$-deep tent $\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}(K)$ over a quasicube K to be the restriction of the tent $\mathbf{T}(K)$ to those points at depth τ or more below K:

$$\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}(K) \equiv \left\{(y,t) \in \mathbf{T}(K) : t \leq 2^{-\tau}\ell(K)\right\}.$$

We can now rewrite the size functional (10.12) of $\mathcal{P}$ as

$$\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})^2 \equiv \sup_{K \in \Pi^{\mathrm{goodbelow}}\mathcal{P}} \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}(K)\right). \tag{10.14}$$

It will be convenient to write

$$\Psi^\alpha(K;\mathcal{P})^2 \equiv \left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}(K)\right),$$

so that we have simply

$$\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})^2 = \sup_{K \in \Pi^{\mathrm{goodbelow}}\mathcal{P}} \frac{\Psi^\alpha(K;\mathcal{P})^2}{|K|_\sigma}.$$

Remark 13. *The functional $\omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}(K)\right)$ is increasing in K, while the functional $\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}$ is 'almost decreasing' in K: if $K_0 \subset K$ then*

$$\begin{aligned}
\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}} &= \int_{A\setminus K} \frac{d\sigma(y)}{\left(|K|^{\frac{1}{n}} + |y - c_K|\right)^{n+1-\alpha}} \\
&\lesssim \int_{A\setminus K} \frac{\left(\sqrt{n}\right)^{n+1-\alpha} d\sigma(y)}{\left(|K_0|^{\frac{1}{n}} + |y - c_{K_0}|\right)^{n+1-\alpha}} \\
&\leq C_{n,\alpha}\int_{A\setminus K_0} \frac{d\sigma(y)}{\left(|K_0|^{\frac{1}{n}} + |y - c_{K_0}|\right)^{n+1-\alpha}} = C_{n,\alpha}\frac{\mathrm{P}^\alpha\left(K_0, \mathbf{1}_{A\setminus K_0}\sigma\right)}{|K_0|^{\frac{1}{n}}},
\end{aligned}$$

since $|K_0|^{\frac{1}{n}} + |y - c_{K_0}| \leq |K|^{\frac{1}{n}} + |y - c_K| + \frac{1}{2}\operatorname{diam}(K)$ for $y \in A \setminus K$.

10.4. **The recursion.** Recall that if $\mathcal{P}$ is an admissible collection for a dyadic quasicube A, the corresponding sublinear form in (10.7) is given in (10.6) by

$$\begin{aligned} \left|\mathsf{B}\right|_{\text{stop},1,\triangle^{\omega}}^{A,\mathcal{P}}(f,g) &\equiv \sum_{J\in\Pi_2\mathcal{P}} \frac{\mathrm{P}^{\alpha}\left(J,\left|\varphi_J^{\mathcal{P}}\right|\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma\right)}{\left|J\right|^{\frac{1}{n}}} \left\|\triangle_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)} \left\|\triangle_J^{\omega}g\right\|_{L^2(\omega)}; \\ \text{where } \varphi_J^{\mathcal{P}} &\equiv \sum_{I\in\mathcal{C}_A':\ (I,J)\in\mathcal{P}} \mathbb{E}_I^{\sigma}\left(\triangle_{\pi I}^{\sigma}f\right)\ \mathbf{1}_{A\setminus I}\ . \end{aligned}$$

In the notation for $\left|\mathsf{B}\right|_{\text{stop},1,\triangle^{\omega}}^{A,\mathcal{P}}$, we are omitting dependence on the parameter α, and to avoid clutter, we will often do so from now on when the dependence on α is inconsequential. Following Lacey [Lac], we now claim the following proposition, from which we obtain (10.7) as a corollary below. Motivated by the conclusion of Proposition 3, we define the *restricted* norm $\mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{P}}$ of the sublinear form $\left|\mathsf{B}\right|_{\text{stop},1,\triangle^{\omega}}^{A,\mathcal{P}}$ to be the best constant $\mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{P}}$ in the inequality

$$\left|\mathsf{B}\right|_{\text{stop},1,\triangle^{\omega}}^{A,\mathcal{P}}(f,g) \leq \mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{P}} \left(\alpha_{\mathcal{A}}(A)\sqrt{\left|A\right|_{\sigma}} + \left\|f\right\|_{L^2(\sigma)}\right) \left\|g\right\|_{L^2(\omega)},$$

where $f\in L^2(\sigma)$ satisfies $\mathbb{E}_I^{\sigma}\left|f\right| \leq \alpha_{\mathcal{A}}(A)$ for all $I\in\mathcal{C}_A^{\text{good}}$.

Proposition 4. *(This is a variant for sublinear forms of the Size Lemma in Lacey* [Lac]*) Suppose $\varepsilon>0$. Let $\mathcal{P}$ be an* admissible *collection of pairs for a dyadic quasicube A. Then we can decompose $\mathcal{P}$ into two disjoint collections $\mathcal{P}=\mathcal{P}^{big}\dot{\cup}\mathcal{P}^{small}$, and further decompose $\mathcal{P}^{small}$ into pairwise disjoint collections $\mathcal{P}_1^{small},\mathcal{P}_2^{small}...\mathcal{P}_{\ell}^{small}...$ i.e.*

$$\mathcal{P}=\mathcal{P}^{big}\dot{\cup}\left(\dot{\bigcup_{\ell=1}^{\infty}}\mathcal{P}_{\ell}^{small}\right),$$

such that the collections $\mathcal{P}^{big}$ and $\mathcal{P}_{\ell}^{small}$ are admissible and satisfy

$$\sup_{\ell\geq1}\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{P}_{\ell}^{small}\right)^2 \leq \varepsilon\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{P}\right)^2, \tag{10.15}$$

and

$$\mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{P}} \leq C_{\varepsilon}\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{P}\right) + \sqrt{n\boldsymbol{\tau}}\sup_{\ell\geq1}\mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{P}_{\ell}^{small}}\ . \tag{10.16}$$

Corollary 2. *The sublinear stopping form inequality (10.7) holds.*

Proof of the Corollary. Set $\mathcal{Q}^0=\mathcal{P}^A$. Apply Proposition 4 to obtain a subdecomposition $\left\{\mathcal{Q}_{\ell}^1\right\}_{\ell=1}^{\infty}$ of $\mathcal{Q}^0$ such that

$$\begin{aligned} &\mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{Q}^0} \leq C_{\varepsilon}\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{Q}^0\right) + \sqrt{n\boldsymbol{\tau}}\sup_{\ell\geq1}\mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{Q}_{\ell}^1}, \\ &\sup_{\ell\geq1}\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{Q}_{\ell}^1\right) \leq \varepsilon\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{Q}^0\right). \end{aligned}$$

Now apply Proposition 4 to each $\mathcal{Q}_{\ell}^1$ to obtain a subdecomposition $\left\{\mathcal{Q}_{\ell,k}^2\right\}_{k=1}^{\infty}$ of $\mathcal{Q}_{\ell}^1$ such that

$$\begin{aligned} \mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{Q}_{\ell}^1} &\leq C_{\varepsilon}\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{Q}_{\ell}^1\right) + \sqrt{n\boldsymbol{\tau}}\sup_{k\geq1}\mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{Q}_{\ell,k}^2}, \\ &\sup_{k\geq1}\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{Q}_{\ell,k}^2\right) \leq \varepsilon\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{Q}_{\ell}^1\right). \end{aligned}$$

Altogether we have

$$\begin{aligned} \mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{Q}^0} &\leq C_{\varepsilon}\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{Q}^0\right) + \sqrt{n\boldsymbol{\tau}}\sup_{\ell\geq1}\left\{C_{\varepsilon}\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{Q}_{\ell}^1\right) + \sqrt{n\boldsymbol{\tau}}\sup_{k\geq1}\mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{Q}_{\ell,k}^2}\right\} \\ &= C_{\varepsilon}\left\{\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{Q}^0\right) + \sqrt{n\boldsymbol{\tau}}\varepsilon\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{Q}^0\right)\right\} + (n\boldsymbol{\tau})\sup_{\ell,k\geq1}\mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{Q}_{\ell,k}^2}\ . \end{aligned}$$

Then with $\zeta \equiv \sqrt{n\boldsymbol{\tau}}$, we obtain by induction for every $N \in \mathbb{N}$,

$$\begin{aligned}\mathfrak{N}_{\operatorname{stop},1,\triangle}^{A,\mathcal{Q}^0} &\leq C_{\varepsilon}\left\{\mathcal{S}_{\operatorname{size}}^{\alpha,A}\left(\mathcal{Q}^0\right)+\zeta\varepsilon\mathcal{S}_{\operatorname{size}}^{\alpha,A}\left(\mathcal{Q}^0\right)+\ldots\zeta^N\varepsilon^N\mathcal{S}_{\operatorname{size}}^{\alpha,A}\left(\mathcal{Q}^0\right)\right\} \\ &\quad +\zeta^{N+1}\sup_{m\in\mathbb{N}^{N+1}}\mathfrak{N}_{\operatorname{stop},1,\triangle}^{A,\mathcal{Q}_m^{N+1}}.\end{aligned}$$

Now we may assume the collection $\mathcal{Q}^0=\mathcal{P}^A$ of pairs is finite (simply truncate the corona $\mathcal{C}_A$ and obtain bounds independent of the truncation) and so $\sup_{m\in\mathbb{N}^{N+1}}\mathfrak{N}_{\operatorname{stop},1,\triangle}^{A,\mathcal{Q}_m^{N+1}}=0$ for N large enough. Then we obtain (10.7) if we choose $0<\varepsilon<\frac{1}{1+\zeta}$ and apply Lemma 19. □

Proof of Proposition 4. Recall that the 'size testing collection' of quasicubes $\Pi^{\operatorname{goodbelow}}\mathcal{P}$ is the collection of all *good* subquasicubes of a quasicube in $\Pi\mathcal{P}$. We may assume that $\mathcal{P}$ is a finite collection. Begin by defining the collection $\mathcal{L}_0$ to consist of the *minimal* dyadic quasicubes K in $\Pi^{\operatorname{goodbelow}}\mathcal{P}$ such that

$$\frac{\Psi^{\alpha}\left(K;\mathcal{P}\right)^2}{\left|K\right|_{\sigma}}\geq\varepsilon\mathcal{S}_{\operatorname{size}}^{\alpha,A}\left(\mathcal{P}\right)^2.$$

where we recall that

$$\Psi^{\alpha}\left(K;\mathcal{P}\right)^2\equiv\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{\left|K\right|^{\frac{1}{n}}}\right)^2\omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\operatorname{deep}}\left(K\right)\right).$$

Note that such minimal quasicubes exist when $0<\varepsilon<1$ because $\mathcal{S}_{\operatorname{size}}^{\alpha,A}\left(\mathcal{P}\right)^2$ is the supremum over $K\in\Pi^{\operatorname{goodbelow}}\mathcal{P}$ of $\frac{\Psi^{\alpha}(K;\mathcal{P})^2}{|K|_{\sigma}}$. A key property of the the minimality requirement is that

$$\frac{\Psi^{\alpha}\left(K';\mathcal{P}\right)^2}{\left|K'\right|_{\sigma}}<\varepsilon\mathcal{S}_{\operatorname{size}}^{\alpha,A}\left(\mathcal{P}\right)^2, \tag{10.17}$$

for all $K'\in\Pi^{\operatorname{goodbelow}}\mathcal{P}$ with $K'\subsetneqq K$ and $K\in\mathcal{L}_0$.

We now perform a stopping time argument 'from the bottom up' with respect to the atomic measure $\omega_{\mathcal{P}}$ in the upper half space. This construction of a stopping time 'from the bottom up' is one of two key innovations in Lacey's argument [Lac], the other being the recursion described in Proposition 4.

We refer to $\mathcal{L}_0$ as the initial or level 0 generation of stopping times. Choose $\rho=1+\varepsilon$. We then recursively define a sequence of generations $\{\mathcal{L}_m\}_{m=0}^{\infty}$ by letting $\mathcal{L}_m$ consist of the *minimal* dyadic quasicubes L in $\Pi^{\operatorname{goodbelow}}\mathcal{P}$ that contain a quasicube from some previous level $\mathcal{L}_{\ell}$, $\ell<m$, such that

$$\omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\operatorname{deep}}\left(L\right)\right)\geq\rho\,\omega_{\mathcal{P}}\left(\bigcup_{L'\in\bigcup_{\ell=0}^{m-1}\mathcal{L}_{\ell}:\ L'\subset L}\mathbf{T}^{\boldsymbol{\tau}-\operatorname{deep}}\left(L'\right)\right). \tag{10.18}$$

Since $\mathcal{P}$ is finite this recursion stops at some level M. We then let $\mathcal{L}_{M+1}$ consist of all the maximal quasicubes in $\Pi^{\operatorname{goodbelow}}\mathcal{P}$ that are not already in some $\mathcal{L}_m$. Thus $\mathcal{L}_{M+1}$ will contain either none, some, or all of the maximal quasicubes in $\Pi^{\operatorname{goodbelow}}\mathcal{P}$. We do not of course have (10.18) for $A'\in\mathcal{L}_{M+1}$ in this case, but we do have that (10.18) fails for subquasicubes K of $A'\in\mathcal{L}_{M+1}$ that are not contained in any other $L\in\mathcal{L}_m$, and this is sufficient for the arguments below.

We now define the collections $\mathcal{P}^{small}$ and $\mathcal{P}^{big}$. The collection $\mathcal{P}^{big}$ will consist of those pairs $(I,J)\in\mathcal{P}$ for which there is $L\in\bigcup_{m=0}^{M+1}\mathcal{L}_m$ with $J\Subset_{\tau}L\subset I$, and $\mathcal{P}^{small}$ will consist of the remaining pairs. But a considerable amount of further analysis is required to prove the conclusion of the proposition. First, let $\mathcal{L}\equiv\bigcup_{m=0}^{M+1}\mathcal{L}_m$ be the tree of stopping quasienergy quasicubes defined above. By our construction above, the maximal elements in $\mathcal{L}$ are the maximal quasicubes in $\Pi^{\operatorname{goodbelow}}\mathcal{P}$. For $L\in\mathcal{L}$, denote by $\mathcal{C}_L$ the *corona* associated with L in the tree $\mathcal{L}$,

$$\mathcal{C}_L\equiv\left\{K\in\Omega\mathcal{D}:K\subset L\text{ and there is no }L'\in\mathcal{L}\text{ with }K\subset L'\subsetneqq L\right\},$$

and define the *shifted* corona by

$$\mathcal{C}_L^{\boldsymbol{\tau}-\mathrm{shift}} \equiv \{K \in \mathcal{C}_L : K \Subset_{\boldsymbol{\tau},\varepsilon} L\} \cup \bigcup_{L' \in \mathfrak{C}_{\mathcal{L}}(L)} \{K \in \Omega\mathcal{D} : K \Subset_{\boldsymbol{\tau},\varepsilon} L \text{ and } K \text{ is } \boldsymbol{\tau}\text{-nearby in } L'\}.$$

Now the parameter m in $\mathcal{L}_m$ refers to the level at which the stopping construction was performed, but for $L \in \mathcal{L}_m$, the corona children L' of L are *not* all necessarily in $\mathcal{L}_{m-1}$, but may be in $\mathcal{L}_{m-t}$ for t large. Thus we need to introduce the notion of geometric depth d in the tree $\mathcal{L}$ by defining

$$\begin{aligned}
\mathcal{G}_0 &\equiv \{L \in \mathcal{L} : L \text{ is maximal}\}, \\
\mathcal{G}_1 &\equiv \{L \in \mathcal{L} : L \text{ is maximal wrt } L \subsetneqq L_0 \text{ for some } L_0 \in \mathcal{G}_0\}, \\
&\vdots \\
\mathcal{G}_{d+1} &\equiv \{L \in \mathcal{L} : L \text{ is maximal wrt } L \subsetneqq L_d \text{ for some } L_d \in \mathcal{G}_d\}, \\
&\vdots
\end{aligned}$$

We refer to $\mathcal{G}_d$ as the d^{th} generation of quasicubes in the tree $\mathcal{L}$, and say that the quasicubes in $\mathcal{G}_d$ are at depth d in the tree $\mathcal{L}$. Thus the quasicubes in $\mathcal{G}_d$ are the stopping quasicubes in $\mathcal{L}$ that are d levels in the *geometric* sense below the top level.

Then for $L \in \mathcal{G}_d$ and $t \geq 0$ define

$$\mathcal{P}_{L,t} \equiv \left\{(I,J) \in \mathcal{P} : I \in \mathcal{C}_L \text{ and } J \in \mathcal{C}_{L'}^{\boldsymbol{\tau}-\mathrm{shift}} \text{ for some } L' \in \mathcal{G}_{d+t} \text{ with } L' \subset L\right\}.$$

In particular, $(I,J) \in \mathcal{P}_{L,t}$ implies that I is in the corona $\mathcal{C}_L$, and that J is in a shifted corona $\mathcal{C}_{L'}^{\boldsymbol{\tau}-\mathrm{shift}}$ that is t levels of generation *below* $\mathcal{C}_L$. We emphasize the distinction 'generation' as this refers to the depth rather than the level of stopping construction. For $t=0$ we further decompose $\mathcal{P}_{L,0}$ as

$$\begin{aligned}
\mathcal{P}_{L,0} &= \mathcal{P}_{L,0}^{small} \dot{\cup} \mathcal{P}_{L,0}^{big}; \\
\mathcal{P}_{L,0}^{small} &\equiv \{(I,J) \in \mathcal{P}_{L,0} : I \neq L\}, \\
\mathcal{P}_{L,0}^{big} &\equiv \{(I,J) \in \mathcal{P}_{L,0} : I = L\},
\end{aligned}$$

with one exception: if $L \in \mathcal{L}_{M+1}$ we set $\mathcal{P}_{L,0}^{small} \equiv \mathcal{P}_{L,0}$ since in this case L fails to satisfy (10.18) as pointed out above. Then we set

$$\begin{aligned}
\mathcal{P}^{big} &\equiv \left\{\bigcup_{L \in \mathcal{L}} \mathcal{P}_{L,0}^{big}\right\} \bigcup \left\{\bigcup_{t \geq 1} \bigcup_{L \in \mathcal{L}} \mathcal{P}_{L,t}\right\}; \\
\left\{\mathcal{P}_\ell^{small}\right\}_{\ell=0}^\infty &\equiv \left\{\mathcal{P}_{L,0}^{small}\right\}_{L \in \mathcal{L}}, \qquad \text{after relabelling.}
\end{aligned}$$

It is important to note that by (10.11), every pair $(I,J) \in \mathcal{P}$ will be included in either $\mathcal{P}^{small}$ or $\mathcal{P}^{big}$. Now we turn to proving the inequalities (10.15) and (10.16).

To prove the inequality (10.15), it suffices with the above relabelling to prove the following claim:

$$\mathcal{S}_{\mathrm{size}}^{\alpha,A}\left(\mathcal{P}_{L,0}^{small}\right)^2 \leq (\rho - 1)\, \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})^2, \qquad L \in \mathcal{L}. \tag{10.19}$$

To see (10.19), suppose first that $L \notin \mathcal{L}_{M+1}$. In the case that $L \in \mathcal{L}_0$ is an initial generation quasicube, then from (10.17) we obtain that

$$\begin{aligned}
&\mathcal{S}_{\mathrm{size}}^{\alpha,A}\left(\mathcal{P}_{L,0}^{small}\right)^2 \\
\leq &\sup_{K' \in \Pi^{\mathrm{goodbelow}}\mathcal{P}:\ K' \subsetneqq L} \frac{1}{|K'|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K', \mathbf{1}_{A \setminus K'}\sigma\right)}{|K'|^{\frac{1}{n}}}\right)^2 \omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}(K')\right) \leq \varepsilon \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})^2.
\end{aligned}$$

Now suppose that $L \notin \mathcal{L}_0$ and also that $L \notin \mathcal{L}_{M+1}$. Pick a pair $(I,J) \in \mathcal{P}_{L,0}^{small}$. Then I is in the restricted corona $\mathcal{C}'_L$ and J is in the $\boldsymbol{\tau}$-*shifted* corona $\mathcal{C}_L^{\boldsymbol{\tau}-\mathrm{shift}}$. Since $\mathcal{P}_{L,0}^{small}$ is a finite collection, the definition of $\mathcal{S}_{\mathrm{size}}^{\alpha,A}\left(\mathcal{P}_{L,0}^{small}\right)$ shows that there is a quasicube $K \in \Pi^{\mathrm{goodbelow}}\mathcal{P}_{L,0}^{small}$ so that

$$\mathcal{S}_{\mathrm{size}}^{\alpha,A}\left(\mathcal{P}_{L,0}^{small}\right)^2 = \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A \setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}(K)\right).$$

Now define

$$t' = t'(K) \equiv \max\{s : \text{there is } L' \in \mathcal{L}_s \text{ with } L' \subset K\}.$$

First, suppose that $t' = 0$ so that K does not contain any $L' \in \mathcal{L}$. Then it follows from our construction at level $\ell = 0$ that

$$\frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A \setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(K)\right) < \varepsilon \mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2,$$

and hence from $\rho = 1 + \varepsilon$ we obtain

$$\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{P}_{L,0}^{small}\right)^2 < \varepsilon \mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2 = (\rho - 1)\,\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2.$$

Now suppose that $t' \geq 1$. Then K fails the stopping condition (10.18) with $m = t' + 1$, and so

$$\omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(K)\right) < \rho\,\omega_{\mathcal{P}}\left(\bigcup_{L' \in \bigcup_{\ell=0}^{t'} \mathcal{L}_\ell :\ L' \subset K} \mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(L')\right).$$

Now we use the crucial fact that $\omega_{\mathcal{P}}$ is *additive* and finite to obtain from this that

$$\begin{aligned}
&\omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(K) \setminus \bigcup_{L' \in \bigcup_{\ell=0}^{t'} \mathcal{L}_\ell :\ L' \subset K} \mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(L')\right) \\
&= \omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(K)\right) - \omega_{\mathcal{P}}\left(\bigcup_{L' \in \bigcup_{\ell=0}^{t'} \mathcal{L}_\ell :\ L' \subset K} \mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(L')\right) \\
&\leq (\rho - 1)\,\omega_{\mathcal{P}}\left(\bigcup_{L' \in \bigcup_{\ell=0}^{t'} \mathcal{L}_\ell :\ L' \subset K} \mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(L')\right).
\end{aligned} \tag{10.20}$$

Thus using

$$\omega_{\mathcal{P}_{L,0}^{small}}\left(\mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(K)\right) \leq \omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(K) \setminus \bigcup_{L' \in \bigcup_{\ell=0}^{t'} \mathcal{L}_\ell :\ L' \subset K} \mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(L')\right),$$

and (10.20) we have

$$\begin{aligned}
&\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{P}_{L,0}^{small}\right)^2\\
\leq\ & \sup_{K\in\Pi^{\text{goodbelow}}\mathcal{P}_{L,0}^{small}} \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(K)\setminus \bigcup_{L'\in\bigcup_{\ell=0}^{t'}\mathcal{L}_\ell:\ L'\subset K}\mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(L')\right)\\
\leq\ & (\rho-1)\sup_{K\in\Pi^{\text{goodbelow}}\mathcal{P}_{L,0}^{small}} \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \omega_{\mathcal{P}}\left(\bigcup_{L'\in\bigcup_{\ell=0}^{t'}\mathcal{L}_\ell:\ L'\subset K}\mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(L')\right).
\end{aligned}$$

and we can continue with

$$\begin{aligned}
&\mathcal{S}_{\text{size}}^{\alpha,A}\left(\mathcal{P}_{L,0}^{small}\right)\\
\leq\ & (\rho-1)\sup_{K\in\Pi^{\text{goodbelow}}\mathcal{P}} \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(K)\right)\\
\leq\ & (\rho-1)\,\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2.
\end{aligned}$$

In the remaining case where $L\in\mathcal{L}_{M+1}$ we can include L as a testing quasicube K and the same reasoning applies. This completes the proof of (10.19).

To prove the other inequality (10.16), we need a lemma to bound the norm of certain 'straddled' stopping forms by the size functional $\mathcal{S}_{\text{size}}^{\alpha,A}$, and another lemma to bound sums of 'mutually orthogonal' stopping forms. We interrupt the proof to turn to these matters. ☐

10.4.1. *The Straddling Lemma.* Given an admissible collection of pairs $\mathcal{Q}$ for A, and a subpartition $\mathcal{S}\subset\Pi^{\text{goodbelow}}\mathcal{Q}$ of pairwise disjoint quasicubes in A, we say that $\mathcal{Q}$ *$\boldsymbol{\tau}$-straddles* $\mathcal{S}$ if for every pair $(I,J)\in\mathcal{Q}$ there is $S\in\mathcal{S}\cap[J,I]$ where $[J,I]$ denotes the geodesic in the dyadic tree $\Omega\mathcal{D}$ that connects J to I, and moreover that $J\Subset_{\boldsymbol{\tau},\varepsilon}S$. Denote by $\mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}^{\text{good}}(S)$ the finite collection of quasicubes that are both good and $(\boldsymbol{\rho}-\boldsymbol{\tau})$-nearby in S. For any good dyadic quasicube $S\in\Omega\mathcal{D}_{\text{good}}$, we will also need the collection $\mathcal{W}^{\text{good}}(S)$ of maximal *good* subquasicubes I of S whose triples $3I$ are contained in S.

Lemma 20. *Let $\mathcal{S}$ be a subpartition of A, and suppose that $\mathcal{Q}$ is an admissible collection of pairs for A such that $\mathcal{S}\subset\Pi^{\text{goodbelow}}\mathcal{Q}$, and such that $\mathcal{Q}$ $\boldsymbol{\tau}$-straddles $\mathcal{S}$. Then we have the sublinear form bound*

$$\mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{Q}}\leq C_{\mathbf{r},\boldsymbol{\tau},\boldsymbol{\rho}}\sup_{S\in\mathcal{S}}\mathcal{S}_{\text{size}}^{\alpha,A;S}(\mathcal{Q})\leq C_{\mathbf{r},\boldsymbol{\tau},\boldsymbol{\rho}}\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{Q}),$$

where $\mathcal{S}_{\text{size}}^{\alpha,A;S}$ is an S-localized version of $\mathcal{S}_{\text{size}}^{\alpha,A}$ with an S-hole given by

$$\mathcal{S}_{\text{size}}^{\alpha,A;S}(\mathcal{Q})^2\equiv\sup_{K\in\mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}^{\text{good}}(S)\cup\mathcal{W}^{\text{good}}(S)}\frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2\omega_{\mathcal{Q}}\left(\mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(K)\right).\tag{10.21}$$

Proof. For $S \in S$ let $\mathcal{Q}^S \equiv \{(I,J) \in \mathcal{Q} : J \Subset_{\boldsymbol{\tau},\varepsilon} S \subset I\}$. We begin by using that $\mathcal{Q}$ $\boldsymbol{\tau}$-straddles $\mathcal{S}$, together with the sublinearity property (10.5) of $\varphi_J^{\mathcal{Q}}$, to write

$$\begin{aligned}
\left|\mathsf{B}\right|_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}}(f,g) &= \sum_{J\in\Pi_2\mathcal{P}} \frac{\mathrm{P}^\alpha\left(J,\left|\varphi_J^{\mathcal{Q}}\right|\mathbf{1}_{A\setminus I_{\mathcal{Q}}(J)}\sigma\right)}{|J|^{\frac{1}{n}}}\left\|\triangle_J^\omega \mathbf{x}\right\|_{L^2(\omega)}\left\|\triangle_J^\omega g\right\|_{L^2(\omega)}\\
&\leq \sum_{S\in\mathcal{S}}\sum_{J\in\Pi_2^{S,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{Q}} \frac{\mathrm{P}^\alpha\left(J,\left|\varphi_J^{\mathcal{Q}^S}\right|\mathbf{1}_{A\setminus I_{\mathcal{Q}}(J)}\sigma\right)}{|J|^{\frac{1}{n}}}\left\|\triangle_J^\omega \mathbf{x}\right\|_{L^2(\omega)}\left\|\triangle_J^\omega g\right\|_{L^2(\omega)};\\
\text{where } \varphi_J^{\mathcal{Q}^S} &\equiv \sum_{I\in\Pi_1\mathcal{Q}^S:\ (I,J)\in\mathcal{Q}^S} \mathbb{E}_I^\sigma\left(\triangle_{\pi I}^\sigma f\right)\ \mathbf{1}_{A\setminus I}\ .
\end{aligned}$$

At this point, with S fixed for the moment, we consider separately the finitely many cases $\ell(J) = 2^{-s}\ell(S)$ where $s \geq \boldsymbol{\rho}$ and where $\boldsymbol{\tau} \leq s < \boldsymbol{\rho}$. More precisely, we pigeonhole the side length of $J \in \Pi_2\mathcal{Q}^S = \Pi_2^{S,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{Q}$ by

$$\begin{aligned}
\mathcal{Q}_*^S &\equiv \left\{(I,J)\in\mathcal{Q}^S : J\in\Pi_2\mathcal{Q}^S \text{ and } \ell(J)\leq 2^{-\boldsymbol{\rho}}\ell(S)\right\},\\
\mathcal{Q}_s^S &\equiv \left\{(I,J)\in\mathcal{Q}^S : J\in\Pi_2\mathcal{Q}^S \text{ and } \ell(J)= 2^{-s}\ell(S)\right\},\qquad \boldsymbol{\tau}\leq s<\boldsymbol{\rho}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\Pi_2\mathcal{Q}_*^S &\equiv \left\{J\in\Pi_2\mathcal{Q}^S : \ell(J)\leq 2^{-\boldsymbol{\rho}}\ell(S)\right\},\\
\Pi_2\mathcal{Q}_s^S &\equiv \left\{J\in\Pi_2\mathcal{Q}^S : \ell(J)= 2^{-s}\ell(S)\right\},\qquad \boldsymbol{\tau}\leq s<\boldsymbol{\rho},
\end{aligned}$$

and we make the corresponding decomposition for the sublinear form

$$\begin{aligned}
\left|\mathsf{B}\right|_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}}(f,g) &= \left|\mathsf{B}\right|_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_*}(f,g)+\sum_{\boldsymbol{\tau}\leq s<\boldsymbol{\rho}}\left|\mathsf{B}\right|_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_s}(f,g)\\
&\equiv \sum_{S\in\mathcal{S}}\sum_{J\in\Pi_2\mathcal{Q}_*^S} \frac{\mathrm{P}^\alpha\left(J,\left|\varphi_J^{\mathcal{Q}_*^S}\right|\mathbf{1}_{A\setminus I_{\mathcal{Q}_*}(J)}\sigma\right)}{|J|^{\frac{1}{n}}}\left\|\triangle_J^\omega \mathbf{x}\right\|_{L^2(\omega)}\left\|\triangle_J^\omega g\right\|_{L^2(\omega)}\\
&\quad+\sum_{\boldsymbol{\tau}\leq s<\boldsymbol{\rho}}\sum_{S\in\mathcal{S}}\sum_{J\in\Pi_2\mathcal{Q}_s^S} \frac{\mathrm{P}^\alpha\left(J,\left|\varphi_J^{\mathcal{Q}_s^S}\right|\mathbf{1}_{A\setminus I_{\mathcal{Q}_S}(J)}\sigma\right)}{|J|^{\frac{1}{n}}}\left\|\triangle_J^\omega \mathbf{x}\right\|_{L^2(\omega)}\left\|\triangle_J^\omega g\right\|_{L^2(\omega)}\ .
\end{aligned}$$

By the tree-connected property of $\mathcal{Q}$, and the telescoping property of martingale differences, together with the bound $\alpha_{\mathcal{A}}(A)$ on the averages of f in the corona $\mathcal{C}_A$, we have

$$\left|\varphi_J^{\mathcal{Q}_*^S}\right|,\left|\varphi_J^{\mathcal{Q}_s^S}\right| \lesssim \alpha_{\mathcal{A}}(A)\,1_{A\setminus I_{\mathcal{Q}^S}(J)}, \tag{10.22}$$

where $I_{\mathcal{Q}^S}(J) \equiv \bigcap\left\{I : (I,J)\in\mathcal{Q}^S\right\}$ is the smallest quasicube I for which $(I,J)\in\mathcal{Q}^S$.

Case for $\left|\mathsf{B}\right|_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_s^S}(f,g)$ when $\boldsymbol{\tau}\leq s\leq\boldsymbol{\rho}$: Now is a crucial definition that permits us to bound the form by the size functional with a large hole. Let

$$\mathcal{C}_s^S \equiv \pi^{\boldsymbol{\tau}}\left(\Pi_2\mathcal{Q}_s^S\right)$$

be the collection of $\boldsymbol{\tau}$-parents of quasicubes in $\Pi_2\mathcal{Q}_s^S$, and denote by $\mathcal{M}_s^S$ the set of *maximal* quasicubes in the collection $\mathcal{C}_s^S$. We have that the quasicubes in $\mathcal{M}_s^S$ are good by our assumption that the quasiHaar support of g is contained in the $\boldsymbol{\tau}$-good quasigrid $\Omega\mathcal{D}^{\boldsymbol{\tau}}_{(\mathbf{r},\varepsilon)-\mathrm{good}}$, and so $\mathcal{M}_s^S\subset\mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}(S)$. Here is the first of two key inclusions:

$$J\Subset_{\boldsymbol{\tau},\varepsilon} K\subset S \text{ if } K\in\mathcal{M}_s^S \text{ is the unique quasicube containing } J. \tag{10.23}$$

Let $I_s\equiv\pi^{\boldsymbol{\rho}-s}S$ so that for each J in $\Pi_2\mathcal{Q}_s^S$ we have the second key inclusion

$$\pi^{\boldsymbol{\rho}}J = I_s\subset I_{\mathcal{Q}^S}(J)\,. \tag{10.24}$$

Now each $K\in\mathcal{M}_s^S$ is also $(\boldsymbol{\rho}-\boldsymbol{\tau})$-deeply embedded in I_s if $\boldsymbol{\rho}\geq\mathbf{r}+\boldsymbol{\tau}$, so that in particular, $3K\subset I_s$. This and (10.24) have the consequence that the following Poisson inequalities hold:

$$\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{A\setminus I_{\mathcal{Q}^S}(J)}\sigma\right)}{|J|^{\frac{m}{n}}}\lesssim\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{A\setminus I_s}\sigma\right)}{|J|^{\frac{m}{n}}}\lesssim\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus I_s}\sigma\right)}{|K|^{\frac{m}{n}}}\lesssim\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{m}{n}}}.$$

Let $\Pi_2\mathcal{Q}_s^S(K)\equiv\left\{J\in\Pi_2\mathcal{Q}_s^S:J\subset K\right\}$. Let

$$\begin{aligned}\left[\Pi_2\mathcal{Q}_s^S\right]_\ell&\equiv\left\{J\in\Pi_2\mathcal{Q}_s^S:\ell(J')=2^{-\ell}\ell(K)\right\},\\ \left[\Pi_2\mathcal{Q}_s^S\right]_\ell^*&\equiv\left\{J':J'\subset J\in\Pi_2\mathcal{Q}_s^S:\ell(J')=2^{-\ell}\ell(K)\right\}.\end{aligned}$$

Now set $\mathcal{Q}_s\equiv\bigcup_{S\in\mathcal{S}}\mathcal{Q}_s^S$. We apply (10.22) and Cauchy-Schwarz in J to bound $|\mathsf{B}|_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_s}(f,g)$ by

$$\alpha_{\mathcal{A}}(A)\sum_{S\in\mathcal{S}}\sum_{K\in\mathcal{M}_s^S}\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{1}{n}}}\right)\left\|\mathsf{P}^{\omega}_{\Pi_2^{S,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{Q}_s;K}\mathbf{x}\right\|_{L^2(\omega)}\left\|\mathsf{P}^{\omega}_{\Pi_2^{S,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{Q}_s;K}g\right\|_{L^2(\omega)},$$

where the localized projections $\mathsf{P}^{\omega}_{\Pi_2^{S,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{Q}_s;K}$ are defined in (8.1) above.

Thus using Cauchy-Schwarz in K we have that $\left|\mathsf{B}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_s}(f,g)\right|$ is bounded by

$$\begin{aligned}&\alpha_{\mathcal{A}}(A)\sum_{S\in\mathcal{S}}\sum_{K\in\mathcal{M}_s^S}\sqrt{|K|_\sigma}\\ &\times\frac{1}{\sqrt{|K|_\sigma}}\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{1}{n}}}\right)\left\|\mathsf{P}^{\omega}_{\Pi_2\mathcal{Q}_s^S(K)}\mathbf{x}\right\|_{L^2(\omega)}\left\|\mathsf{P}^{\omega}_{\Pi_2\mathcal{Q}_s^S(K)}g\right\|_{L^2(\omega)}\\ \leq\;&\alpha_{\mathcal{A}}(A)\sup_{S\in\mathcal{S}}\mathcal{S}_{\mathrm{size}}^{\alpha,A;S}(\mathcal{Q})\left(\sum_{S\in\mathcal{S}}\sum_{K\in\mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}(S)}|K|_\sigma\right)^{\frac{1}{2}}\|g\|_{L^2(\omega)}\\ \leq\;&\sup_{S\in\mathcal{S}}\mathcal{S}_{\mathrm{size}}^{\alpha,A;S}(\mathcal{Q})\,\alpha_{\mathcal{A}}(A)\sqrt{|A|_\sigma}\,\|g\|_{L^2(\omega)},\end{aligned}$$

since $J\Subset_{\boldsymbol{\tau},\varepsilon}M\subset K$ by (10.23), since $\mathcal{M}_s^S\subset\mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}(S)$, and since the collection of quasicubes $\bigcup_{S\in\mathcal{S}}\mathcal{M}_s^S$ is pairwise disjoint in A.

Case for $|\mathsf{B}|_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_*}(f,g)$: This time we let $\mathcal{C}_*^S\equiv\pi^{\boldsymbol{\tau}}\left(\Pi_2\mathcal{Q}_*^S\right)$ and denote by $\mathcal{M}_*^S$ the set of *maximal* quasicubes in the collection $\mathcal{C}_*^S$. We have the two key inclusions,

$$J\Subset_{\boldsymbol{\tau},\varepsilon}M\Subset_{\boldsymbol{\rho}-\boldsymbol{\tau},\varepsilon}S\text{ if }M\in\mathcal{M}_*^S\text{ is the unique quasicube containing }J,$$

and

$$\pi^{\boldsymbol{\rho}}J\subset S\subset I_{\mathcal{Q}}(J).$$

Moreover there is $K\in\mathcal{W}^{\mathrm{good}}(S)$ that contains M. Thus $3K\subset S$ and we have

$$\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{A\setminus S}\sigma\right)}{|J|^{\frac{1}{n}}}\lesssim\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{1}{n}}},$$

and $|\varphi_J| \lesssim \alpha_{\mathcal{A}}(A) 1_{A\setminus S}$. Now set $\mathcal{Q}_* \equiv \bigcup_{S\in\mathcal{S}} \mathcal{Q}_*^S$. Arguing as above, but with $\mathcal{W}^{\text{good}}(S)$ in place of $\mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}(S)$, and using $J \Subset_{\boldsymbol{\rho},\varepsilon} I_{\mathcal{Q}}(J)$, we can bound $|\mathsf{B}|^{A,\mathcal{Q}_*}_{\text{stop},1,\triangle}(f,g)$ by

$$\begin{aligned}
& \alpha_{\mathcal{A}}(A) \sum_{S\in\mathcal{S}} \sum_{K\in\mathcal{W}^{\text{good}}(S)} \sqrt{|K|_\sigma} \\
& \times \frac{1}{\sqrt{|K|_\sigma}} \left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{1}{n}}} \right) \left\| \mathsf{P}^\omega_{\Pi_2 \mathcal{Q}^S_s(K)} \mathbf{x} \right\|_{L^2(\omega)} \left\| \mathsf{P}^\omega_{\Pi_2 \mathcal{Q}^S_*(K)} g \right\|_{L^2(\omega)} \\
\leq \; & \alpha_{\mathcal{A}}(A) \sup_{S\in\mathcal{S}} \mathcal{S}^{\alpha,A;S}_{\text{size}}(\mathcal{Q}) \left(\sum_{S\in\mathcal{S}} \sum_{K\in\mathcal{W}^{\text{good}}(S)} |K|_\sigma \right)^{\frac{1}{2}} \|g\|_{L^2(\omega)} \\
\leq \; & \sup_{S\in\mathcal{S}} \mathcal{S}^{\alpha,A;S}_{\text{size}}(\mathcal{Q})\, \alpha_{\mathcal{A}}(A) \sqrt{|A|_\sigma} \, \|g\|_{L^2(\omega)} .
\end{aligned}$$

We now sum these bounds in s and $*$ and use $\sup_{S\in\mathcal{S}} \mathcal{S}^{\alpha,A;S}_{\text{size}}(\mathcal{Q}) \leq \mathcal{S}^{\alpha,A}_{\text{size}}(\mathcal{Q})$ to complete the proof of Lemma 20. □

10.4.2. *The Orthogonality Lemma.* Given a set $\{\mathcal{Q}_m\}_{m=0}^\infty$ of admissible collections for A, we say that the collections $\mathcal{Q}_m$ are *mutually orthogonal*, if each collection $\mathcal{Q}_m$ satisfies

$$\mathcal{Q}_m \subset \bigcup_{j=0}^\infty \{\mathcal{A}_{m,j} \times \mathcal{B}_{m,j}\} \ ,$$

where the sets $\{\mathcal{A}_{m,j}\}_{m,j}$ and $\{\mathcal{B}_{m,j}\}_{m,j}$ each have bounded overlap on the dyadic quasigrid $\Omega\mathcal{D}$:

$$\sum_{m,j=0}^\infty \mathbf{1}_{\mathcal{A}_{m,j}} \leq A \mathbf{1}_{\Omega\mathcal{D}} \text{ and } \sum_{m,j=0}^\infty \mathbf{1}_{\mathcal{B}_{m,j}} \leq B \mathbf{1}_{\Omega\mathcal{D}}.$$

Lemma 21. *Suppose that $\{\mathcal{Q}_m\}_{m=0}^\infty$ is a set of admissible collections for A that are* mutually orthogonal*. Then if $\mathcal{Q} \equiv \bigcup_{m=0}^\infty \mathcal{Q}_m$, the sublinear stopping form $|\mathsf{B}|^{A,\mathcal{Q}}_{\text{stop},1,\triangle}(f,g)$ has its restricted norm $\mathfrak{N}^{A,\mathcal{Q}}_{\text{stop},1,\triangle}$ controlled by the* supremum *of the restricted norms $\mathfrak{N}^{A,\mathcal{Q}_m}_{\text{stop},1,\triangle}$:*

$$\mathfrak{N}^{A,\mathcal{Q}}_{\text{stop},1,\triangle} \leq \sqrt{nAB} \sup_{m\geq 0} \mathfrak{N}^{A,\mathcal{Q}_m}_{\text{stop},1,\triangle}.$$

Proof. If $\mathsf{P}^\sigma_m = \sum_{j\geq 0} \sum_{I\in\mathcal{A}_{m,j}} \triangle^\sigma_{\pi I}$ (note the parent πI in the projection $\triangle^\sigma_{\pi I}$ because of our 'change of dummy variable' in (10.1)) and $\mathsf{P}^\omega_m = \sum_{j\geq 0} \sum_{J\in\mathcal{B}_{m,j}} \triangle^\omega_J$, then we have

$$\mathsf{B}^{A,\mathcal{Q}_m}_{\text{stop}}(f,g) = \mathsf{B}^{A,\mathcal{Q}_m}_{\text{stop}}\left(\mathsf{P}^\sigma_m f, \mathsf{P}^\omega_m g\right),$$

and

$$\begin{aligned}
\sum_{m\geq 0} \|\mathsf{P}^\sigma_m f\|^2_{L^2(\sigma)} &\leq \sum_{m\geq 0}\sum_{j\geq 0} \left\| \mathsf{P}^\sigma_{\mathcal{A}_{m,j}} f \right\|^2_{L^2(\sigma)} \leq An \|f\|^2_{L^2(\sigma)} , \\
\sum_{m\geq 0} \|\mathsf{P}^\omega_m g\|^2_{L^2(\sigma)} &\leq \sum_{m\geq 0}\sum_{j\geq 0} \left\| \mathsf{P}^\omega_{\mathcal{B}_{m,j}} g \right\|^2_{L^2(\omega)} \leq B \|g\|^2_{L^2(\omega)} \ .
\end{aligned}$$

The sublinear inequality (10.5) and Cauchy-Schwarz now give

$$\begin{aligned}
|\mathsf{B}|^{A,\mathcal{Q}}_{\text{stop},1,\triangle}(f,g) &\leq \sum_{m\geq 0}|\mathsf{B}|^{A,\mathcal{Q}_m}_{\text{stop},1,\triangle}(f,g)\leq \sum_{m\geq 0}\mathfrak{N}^{A,\mathcal{Q}_m}_{stop}\left\Vert \mathsf{P}^{\sigma}_m f\right\Vert_{L^2(\sigma)}\left\Vert \mathsf{P}^{\omega}_m g\right\Vert_{L^2(\sigma)}\\
&\leq \left(\sup_{m\geq 0}\mathfrak{N}^{A,\mathcal{Q}_m}_{\text{stop},1,\triangle}\right)\sqrt{\sum_{m\geq 0}\left\Vert \mathsf{P}^{\sigma}_m f\right\Vert^2_{L^2(\sigma)}}\sqrt{\sum_{m\geq 0}\left\Vert \mathsf{P}^{\omega}_m g\right\Vert^2_{L^2(\sigma)}}\\
&\leq \left(\sup_{m\geq 0}\mathfrak{N}^{A,\mathcal{Q}_m}_{\text{stop},1,\triangle}\right)\sqrt{nAB}\sqrt{n}\left\Vert f\right\Vert_{L^2(\sigma)}\left\Vert g\right\Vert_{L^2(\omega)}.
\end{aligned}$$

□

10.4.3. *Completion of the proof.* Now we return to the proof of inequality (10.16) in Proposition 4.

Proof of (10.16). Recall that

$$\begin{aligned}
\mathcal{P}^{big} &= \left\{\bigcup_{L\in\mathcal{L}}\mathcal{P}^{big}_{L,0}\right\}\bigcup\left\{\bigcup_{t\geq 1}\bigcup_{L\in\mathcal{L}}\mathcal{P}_{L,t}\right\}\equiv \mathcal{Q}^{big}_0\bigcup\mathcal{Q}^{big}_1;\\
\mathcal{Q}^{big}_0 &\equiv \bigcup_{L\in\mathcal{L}}\mathcal{P}^{big}_{L,0}\ ,\qquad \mathcal{Q}^{big}_1\equiv\bigcup_{t\geq 1}\mathcal{P}^{big}_t,\qquad \mathcal{P}^{big}_t\equiv\bigcup_{L\in\mathcal{L}}\mathcal{P}_{L,t}.
\end{aligned}$$

We first consider the collection $\mathcal{Q}^{big}_0=\bigcup_{L\in\mathcal{L}}\mathcal{P}^{big}_{L,0}$, and claim that

$$\mathfrak{N}^{A,\mathcal{P}^{big}_{L,0}}_{\text{stop},1,\triangle}\leq C\mathcal{S}^{\alpha,A}_{\text{size}}\left(\mathcal{P}^{big}_{L,0}\right)\leq C\mathcal{S}^{\alpha,A}_{\text{size}}\left(\mathcal{P}\right),\qquad L\in\mathcal{L}. \tag{10.25}$$

To see this we note that $\mathcal{P}^{big}_{L,0}$ $\boldsymbol{\tau}$-straddles the trivial collection $\{L\}$ consisting of a single quasicube, since the pairs (I,J) that arise in $\mathcal{P}^{big}_{L,0}$ have $I=L$ and J in the shifted corona $\mathcal{C}^{\boldsymbol{\tau}-\text{shift}}_I$. Thus we can apply Lemma 20 with $\mathcal{Q}=\mathcal{P}^{big}_{L,0}$ and $\mathcal{S}=\{L\}$ to obtain (10.25).

Next, we observe that the collections $\mathcal{P}^{big}_{L,0}$ are *mutually orthogonal*, namely

$$\begin{aligned}
\mathcal{P}^{big}_{L,0} &\subset \mathcal{C}_L\times\mathcal{C}^{\boldsymbol{\tau}-\text{shift}}_L\ ,\\
\sum_{L\in\mathcal{L}}\mathbf{1}_{\mathcal{C}_L} &\leq 1\text{ and }\sum_{L\in\mathcal{L}}\mathbf{1}_{\mathcal{C}^{\boldsymbol{\tau}-\text{shift}}_L}\leq\boldsymbol{\tau}.
\end{aligned}$$

Thus the Orthogonality Lemma 21 shows that

$$\mathfrak{N}^{A,\mathcal{Q}^{big}_0}_{\text{stop},1,\triangle}\leq\sqrt{n\boldsymbol{\tau}}\sup_{L\in\mathcal{L}}\mathfrak{N}^{A,\mathcal{P}^{big}_{L,0}}_{\text{stop},1,\triangle}\leq\sqrt{n\boldsymbol{\tau}}C\mathcal{S}^{\alpha,A}_{\text{size}}\left(\mathcal{P}\right).$$

Now we turn to the collection

$$\begin{aligned}
\mathcal{Q}^{big}_1 &= \bigcup_{t\geq 1}\bigcup_{L\in\mathcal{L}}\mathcal{P}_{L,t}=\bigcup_{t\geq 1}\mathcal{P}^{big}_t;\\
\mathcal{P}^{big}_t &\equiv \bigcup_{L\in\mathcal{L}}\mathcal{P}_{L,t}\ ,\qquad t\geq 0.
\end{aligned}$$

We claim that

$$\mathfrak{N}^{A,\mathcal{P}^{big}_t}_{\text{stop},1,\triangle}\leq C\rho^{-\frac{t}{2}}\mathcal{S}^{\alpha,A}_{\text{size}}\left(\mathcal{P}\right),\qquad t\geq 1. \tag{10.26}$$

Note that with this claim established, we have

$$\mathfrak{N}^{A,\mathcal{P}^{big}}_{\text{stop},1,\triangle}\leq\mathfrak{N}^{A,\mathcal{Q}^{big}_0}_{\text{stop},1,\triangle}+\mathfrak{N}^{A,\mathcal{Q}^{big}_1}_{\text{stop},1,\triangle}\leq\mathfrak{N}^{A,\mathcal{Q}^{big}_0}_{\text{stop},1,\triangle}+\sum_{t=1}^{\infty}\mathfrak{N}^{A,\mathcal{P}^{big}_t}_{\text{stop},1,\triangle}\leq C_{\boldsymbol{\rho}}\mathcal{S}^{\alpha,A}_{\text{size}}\left(\mathcal{P}\right),$$

which proves (10.16) if we apply the Orthogonal Lemma 21 to the set of collections $\left\{\mathcal{P}^{small}_{L,0}\right\}_{L\in\mathcal{L}}$, which is mutually orthogonal since $\mathcal{P}^{small}_{L,0}\subset\mathcal{C}'_L\times\mathcal{C}^{\boldsymbol{\tau}-\text{shift}}_L$. With this the proof of Proposition 4 is now complete since $\rho=1+\varepsilon$. Thus it remains only to show that (10.26) holds.

The cases $1 \leq t \leq \mathbf{r}+1$ can be handled with relative ease since decay in t is not needed there. Indeed, $\mathcal{P}_{L,t}$ $\boldsymbol{\tau}$-straddles the collection $\mathfrak{C}_{\mathcal{L}}(L)$ of $\mathcal{L}$-children of L, and so the Straddling Lemma applies to give

$$\mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{P}_{L,t}} \leq C\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P}_{L,t}) \leq C\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P}),$$

and then the Orthogonality Lemma 21 applies to give

$$\mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{P}_t^{big}} \leq \sqrt{n\boldsymbol{\tau}}\sup_{L\in\mathcal{L}}\mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{P}_{L,t}} \leq C\sqrt{n\boldsymbol{\tau}}\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P}),$$

since $\{\mathcal{P}_{L,t}\}_{L\in\mathcal{L}}$ is mutually orthogonal as $\mathcal{P}_{L,t} \subset \mathcal{C}_L \times \mathcal{C}_{L'}^{\boldsymbol{\tau}-\mathrm{shift}}$ with $L \in \mathcal{G}_d$ and $L' \in \mathcal{G}_{d+t}$ for depth $d = d(L)$.

Now we consider the case $t \geq \mathbf{r}+2$, where it is essential to obtain decay in t. We again apply Lemma 20 to $\mathcal{P}_{L,t}$ with $\mathcal{S} = \mathfrak{C}_{\mathcal{L}}(L)$, but this time we must use the stronger localized bounds $\mathcal{S}_{\mathrm{size}}^{\alpha,A;S}$ with an S-hole, that give

$$\mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{P}_{L,t}} \leq C\sup_{S\in\mathfrak{C}_{\mathcal{L}}(L)}\mathcal{S}_{\mathrm{size}}^{\alpha,A;S}(\mathcal{P}_{L,t}), \qquad t \geq 0. \tag{10.27}$$

Fix $L \in \mathcal{G}_d$. Now we note that if $J \in \Pi_2^{L,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P}_{L,t}$ then J belongs to the $\boldsymbol{\tau}$-shifted corona $\mathcal{C}_{L^{d+t}}^{\boldsymbol{\tau}-\mathrm{shift}}$ for some quasicube $L^{d+t} \in \mathcal{G}_{d+t}$. Then $\pi^{\boldsymbol{\tau}}J$ is $\boldsymbol{\tau}$ levels above J, hence in the corona $\mathcal{C}_{L^{d+t}}$. This quasicube L^{d+t} lies in some child $S \in \mathcal{S} = \mathfrak{C}_{\mathcal{L}}(L)$. So fix $S \in \mathcal{S}$ and a quasicube $L^{d+t} \in \mathcal{G}_{d+t}$ that is contained in S with $t \geq \mathbf{r}+2$. Now the quasicubes K that arise in the supremum defining $\mathcal{S}_{\mathrm{size}}^{\alpha,A;S}(\mathcal{P}_{L,t})$ in (10.21) belong to either $\mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}(S)$ or $\mathcal{W}^{\mathrm{good}}(S)$. We will consider these two cases separately.

So first suppose that $K \in \mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}(S)$. A simple induction on levels yields

$$\begin{aligned}
\omega_{\mathcal{P}_{L,t}}\left(\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}(K)\right) &= \sum_{\substack{J\in\Pi_2^{S,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P}_{L,t} \\ J\subset K}} \left\|\triangle_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\
&\leq \omega_{\mathcal{P}}\left(\bigcup_{L^{d+t}\in\mathcal{G}_{d+t}:\ L^{d+t}\subset K}\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}\left(L^{d+t}\right)\right) \\
&\leq \frac{1}{\rho}\omega_{\mathcal{P}}\left(\bigcup_{L^{d+t-1}\in\mathcal{G}_{d+t-1}:\ L^{d+t-1}\subset K}\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}\left(L^{d+t-1}\right)\right) \\
&\ \ \vdots \\
&\lesssim \rho^{-(t-\boldsymbol{\rho}-\boldsymbol{\tau})}\omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}(K)\right), \qquad t \geq \boldsymbol{\rho}-\boldsymbol{\tau}+2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2\omega_{\mathcal{P}_{L,t}}\left(\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}(K)\right) \\
\lesssim\ &\rho^{-t}\frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2\omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}(K)\right) \lesssim \rho^{-t}\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})^2.
\end{aligned}$$

Now suppose that $K \in \mathcal{W}^{\mathrm{good}}(S)$ and that $J \in \Pi_2^{S,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P}_{L,t}$ and $J \subset K$. There is a unique quasicube $L^{d+\mathbf{r}+1} \in \mathcal{G}_{d+\mathbf{r}+1}$ such that $J \subset L^{d+\mathbf{r}+1} \subset S$. Now $L^{d+\mathbf{r}+1}$ is good so $L^{d+\mathbf{r}+1} \Subset_{\mathbf{r},\varepsilon} S$. Thus in particular $3L^{d+\mathbf{r}+1} \subset S$ so that $L^{d+\mathbf{r}+1} \subset K$. The above simple induction applies here to give

$$\begin{aligned}
\sum_{\substack{J\in\Pi_2^{S,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P}_{L,t} \\ J\subset L^{d+\mathbf{r}+1}}} \left\|\triangle_J^{\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 &\leq \omega_{\mathcal{P}}\left(\bigcup_{L^{d+t}\in\mathcal{G}_{d+t}:\ L^{m-t}\subset L^{d+\mathbf{r}+1}}\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}\left(L^{d+t}\right)\right) \\
&\lesssim \rho^{-(t-1-\mathbf{r})}\omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}\left(L^{d+\mathbf{r}+1}\right)\right), \qquad t \geq \mathbf{r}+2.
\end{aligned}$$

Thus we have,

$$\begin{aligned}
&\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^{2}\sum_{\substack{J\in\Pi_{2}^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P}_{L,t}\\ J\subset K}}\left\Vert \triangle_{J}^{\omega}\mathbf{x}\right\Vert_{L^{2}(\omega)}^{2}\\
&\leq C\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^{2}\rho^{-(t-1-\mathbf{r})}\sum_{\substack{L^{d+\mathbf{r}+1}\in\mathcal{G}_{d+\mathbf{r}+1}\\ L^{d+\mathbf{r}+1}\subset K}}\omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}\left(L^{d+\mathbf{r}+1}\right)\right)\\
&\leq C\rho^{-(t-1-\mathbf{r})}\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^{2}\omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}-\mathrm{deep}}(K)\right)\leq C\rho^{-(t-1-\mathbf{r})}\mathcal{S}_{\mathrm{size}}^{\alpha,A}\left(\mathcal{P}\right)^{2}.
\end{aligned}$$

So altogether we conclude that

$$\begin{aligned}
&\sup_{S\in\mathfrak{C}_{\mathcal{L}}(L)}\mathcal{S}_{\mathrm{size}}^{\alpha,A;S}\left(\mathcal{P}_{L,t}\right)^{2}\\
&=\sup_{S\in\mathfrak{C}_{\mathcal{L}}(L)}\ \sup_{K\in\mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}(S)\cup\mathcal{W}^{\mathrm{good}}(S)}\frac{1}{|K|_{\sigma}}\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^{2}\sum_{\substack{J\in\Pi_{2}^{K,\boldsymbol{\tau}-\mathrm{deep}}\mathcal{P}_{L,t}\\ J\subset K}}\left\Vert \mathsf{P}_{J}^{\omega}\mathbf{x}\right\Vert_{L^{2}(\omega)}^{2}\\
&\leq C_{\mathbf{r},\boldsymbol{\tau},\boldsymbol{\rho}}\rho^{-t}\mathcal{S}_{\mathrm{size}}^{\alpha,A}\left(\mathcal{P}\right)^{2},
\end{aligned}$$

and combined with (10.27) this gives (10.26). As we pointed out above, this completes the proof of Proposition 4, hence of Proposition 3, and finally of Theorem 1. □

References

[Hyt] Hytönen, Tuomas, *On Petermichl's dyadic shift and the Hilbert transform*, C. R. Math. Acad. Sci. Paris **346** (2008), MR2464252.

[Hyt2] Hytönen, Tuomas, *The two weight inequality for the Hilbert transform with general measures,* *arXiv:1312.0843v2*.

[HyLaPe] Hytönen, Tuomas, Lacey, Michael T., and Pérez, C., *Sharp weighted bounds for the q-variation of singular integrals*, Bull. Lon. Math. Soc. **45** (2013), 529-540.

[Lac] Lacey, Michael T., *On the two weight Hilbert transform inequality*, `arXiv:1301.4663v1`.

[Lac2] Lacey, Michael T., *The two weight inequality for the Hilbert transform: a primer*, `arXiv:1304.5004v1`.

[LaSaUr1] Lacey, Michael T., Sawyer, Eric T., Uriarte-Tuero, Ignacio, *A characterization of two weight norm inequalities for maximal singular integrals,* `arxiv:0807.0246` (2008).

[LaSaUr2] Lacey, Michael T., Sawyer, Eric T., Uriarte-Tuero, Ignacio, *A Two Weight Inequality for the Hilbert transform assuming an energy hypothesis,* `arXiv:` (2011).

[LaSaShUr] Lacey, Michael T., Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, *The Two weight inequality for Hilbert transform, coronas, and energy conditions,* `arXiv:` (2011).

[LaSaShUr2] Lacey, Michael T., Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, *Two Weight Inequality for the Hilbert Transform: A Real Variable Characterization,* `arXiv:1201.4319` (2012).

[LaSaShUr3] Lacey, Michael T., Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, *Two weight inequality for the Hilbert transform: A real variable characterization*, this version submitted to Duke Math. J.

[LaSaShUrWi] Lacey, Michael T., Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, Wick, Brett D., *Two weight inequalities for the Cauchy transform from* $\mathbb{R}$ *to* $\mathbb{C}_{+}$, *arXiv:1310.4820v4*.

[LaWi] Lacey, Michael T., Wick, Brett D., *Two weight inequalities for Riesz transforms: uniformly full dimension weights*, *arXiv:1312.6163v3*.

[NaVo] F. Nazarov and A. Volberg, *The Bellman function, the two weight Hilbert transform, and the embeddings of the model space* K_{θ}, J. d'Analyse Math. **87** (2002), 385-414.

[NTV1] F. Nazarov, S. Treil and A. Volberg, *The Bellman function and two weight inequalities for Haar multipliers*, J. Amer. Math. Soc. **12** (1999), 909-928, MR{1685781 (2000k:42009)}.

[NTV2] Nazarov, F., Treil, S. and Volberg, A., *The Tb-theorem on non-homogeneous spaces,* Acta Math. **190** (2003), no. 2, MR 1998349 (2005d:30053).

[NTV4] F. Nazarov, S. Treil and A. Volberg, *Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures*, preprint (2004) `arxiv:1003.1596`

[Saw] E. Sawyer, *A characterization of two weight norm inequalities for fractional and Poisson integrals*, Trans. A.M.S. **308** (1988), 533-545, MR{930072 (89d:26009)}.

[Expanded] E. Sawyer, An expanded version of '*Two Weight Inequality for the Hilbert Transform: A Real Variable Characterization, arXiv:1201.4319.' posted on the author's website http://www.math.mcmaster.ca/~sawyer/ (2012).*

[SaShUr] Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A *two weight theorem for α-fractional singular integrals: an expanded version*, arXiv:1302.5093v6.

[SaShUr2] Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A *two weight theorem for α-fractional singular integrals with an energy side condition*, arXiv:1302.5093v8.

[SaShUr3] Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, *A geometric condition, necessity of energy, and two weight boundedness of fractional Riesz transforms*, arXiv:1310.4484v3.

[SaShUr4] Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, *A note on failure of energy reversal for classical fractional singular integrals*, arXiv:1401.0467v6.

[SaShUr5] Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, A *two weight theorem for α-fractional singular integrals with an energy side condition and quasicube testing*, arXiv:1302.5093v10.

[SaShUr8] Sawyer, Eric T., Shen, Chun-Yen, Uriarte-Tuero, Ignacio, The *two weight $T1$ theorem for fractional Riesz transforms when one measure is supported on a curve*, arXiv:1505.07822v4.

[SaWh] E. Sawyer and R. L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces, *Amer. J. Math.* **114** (1992), 813-874.

[Ste] E. M. Stein, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, N. J., 1993.

[Vol] A. Volberg, *Calderón-Zygmund capacities and operators on nonhomogeneous spaces,* CBMS Regional Conference Series in Mathematics (2003), MR{2019058 (2005c:42015)}.

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